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JAN 79 Y V CHUYEV , G P SPEKHOVA
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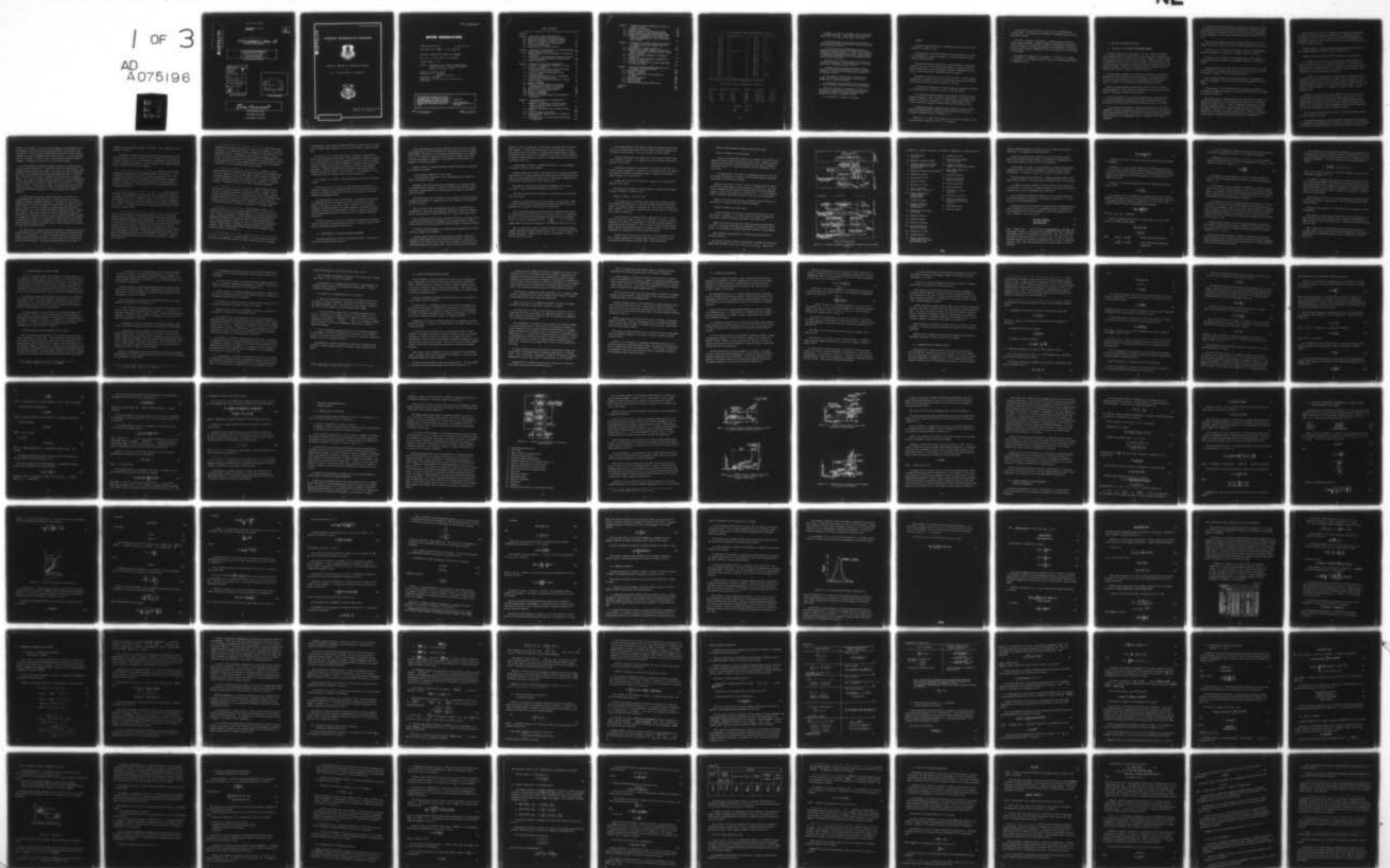

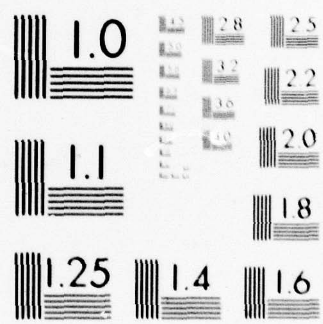

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FOREIGN TECHNOLOGY DIVISION

TECHNICAL PROBLEMS OF OPERATIONS RESEARCH

by

Yu. V. Chuyev and G. P. Spekhova

FTD-ID(RS)T-1986-78

EDITED TRANSLATION

FTD-ID(RS)T-1986-78 22 January 1979

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TECHNICAL PROBLEMS OF OPERATIONS RESEARCH

By: Yu. V. Chuyev and G. P. Spekhova

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TABLE OF CONTENTS

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

Russian	English	Russian	English	Russian	English
sin	sin	sh	sinh	arc sh	$\sinh^{-1}$
cos	cos	ch	cosh	arc ch	$\cosh^{-1}$
tg	tan	th	tanh	arc th	$\tanh^{-1}$
ctg	cot	cth	coth	arc cth	$\coth^{-1}$
sec	sec	sch	sech	arc sch	sech^{-1}
cosec	csc	csch	csch	arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

Chuyev, Yu. V. and G. P. Spekhova. Technical Problems of Operations Research. Moscow, Izdatel'stvo "Sovetskoye Radio," 1971, 244 pages, 14,000 copies. cost 77 kop.

The book describes the applications of operations research methods to solution of problems arising in the designing of any technical devices.

It briefly states the methodological principles of operations research and questions of predicting the characteristics of technical devices. It gives a review of methods of mathematical programming (optimization), from classical to those developed in recent years, and recommends suitable areas for their application.

The main part of the book is devoted to a description of methods and examples of solving three groups of problems which arise in designing technical devices:

-- problem of the replacement of technical devices, the solution of which answers the question of the general necessity of developing a new technical device and when it is suitable to do so;

-- the problem of the selection of optimal series (types), the solution of which answers the question of optimal types of technical devices;

-- the problem of the selection of optimal parameters of technical devices, including reliability and durability.

The book is written for a wide range of readers connected with the designing of technical devices: executives of scientific research institutes and design bureaus, engineers and students of technical schools.

17 illustrations, 31 tables, 55 references.

FOREWORD

At present operations research is finding wide application in the most varied areas of human activity.

The application of operations research to military problems, as well as to the management of industry and commerce, has been mentioned in several previously published works.

There are also enormous possibilities in the application of operations research to the area of the design and development of technical devices, by which here and later we mean any equipment, machinery, mechanisms or instruments. However, there are no summarizing works in this area.

The purpose of this book is to try to fill this void. It gives general propositions of operations research in relation to solving technical problems and discusses three basic groups of problems in this area:

-- the problem of replacement of technical devices, answering the question, is it necessary at all to develop some type of technical device;

-- the problem of selecting optimal series (types) of technical devices, making it possible to select their basic characteristics (for example, flight distance of airplanes, load capacity of trucks, response speed and memory volume of electronic computers, effective range of radio stations, etc.);

-- generalized problems of optimum design, making it possible to select particular characteristics of technical devices (reliability, durability, accuracy of individual devices, throughput, etc.).

Chapters 1, 3, 5, 6 and 7 were written by Yu. V. Chuyev, Chapters 2, 4 and certain sections of Chapters 6 and 7 by G. P. Spekhova.

As this work is one of the first in this field, it undoubtedly also has some faults. The authors will be thankful to readers who send their comments about the book.

The book is intended for executives of scientific-research institutes, design bureaus and industry, engineers engaged in designing any technical devices, and students of technical institutes, as the methodology of operations research is quite general. Mathematicians will be interested in the statement of the problems.

The authors are indebted to D. M. Komarov, V. I. Kuz'min, R. D. Kogan, N. N. Moyseyev, L. N. Presnukhin and Ya. B. Shor, whose valuable advice helped in preparing the book.

1. BRIEF NOTES ON OPERATIONS RESEARCH

1.1. THE SUBJECT AND PROBLEMS OF OPERATIONS RESEARCH

Operations research is a young science. The term "operations research" itself appeared during the Second World War, when joint groups of scientists of various specialties were created to aid the military command in solving a number of complex military-technical and tactical problems. Such problems, as a rule, were connected with the search for efficient methods of using new military technology, development of methods of combatting new kinds of weapons used by the enemy and fast reaction to change in the enemy's tactics.

Military experts, engineers of various skills, physicists, economists and mathematicians were enlisted to solve these problems. The groups were given the task of preparing scientifically based suggestions on some question or other for a report to the appropriate department making the decision.

Thus, the term "operations research" was born with a military overtone, although operations at that time were also understood in the broad sense of the word, i.e. ultimately, purposeful human activity.

It must be pointed out that operations researchers are scientists who do not have any administrative authority and prepare optimal decisions on the instructions of those who make the decision (operating aspect), to aid them.

In the post-war period methods of operations research began to be used increasingly in industry, economics and agriculture, as a result of which they are being improved and generalized. The science of operations research is gradually being formulated into an independent section of cybernetics--the science of management.

At present, by operations research is understood a science whose essence is the use of scientific principles, methods and means for problems connected with the function of systems, in order to give the person who is responsible for the management of given systems optimal solutions for the stated problems [11].

Situations are considered in which a compromise solution must be used, taking into account connections between the studied phenomenon and the others.

A certain step in the function of a system, limited to a fixed segment of time and performance of a fixed task, is commonly called an operation.

An operation can be studied by various means: study of the original of an operation or study of a model of the operation (physical, mathematical or mixed).

Operations research, as a rule, must be conducted with the aid of a model of the operation, as organization and performance of the operation itself for the purpose of studying it is complicated and expensive.

Thus, operations research presupposes construction of a mathematical model of the operation. This in no way indicates any disregard of full-scale studies.

On the contrary, full-scale studies must of necessity be conducted on a scale which will ensure construction of a model and introduction into the model of necessary information.

The ultimate aim of operations research is to find optimal solutions. In talking about optimality, we must indicate what we mean here. In operations research an optimal solution (control) minimizing or maximizing a certain criterion (purposeful function, function of quality) with a given system of limitations. Criterion K is a certain function (functional) of a solution (control $\overline{U}$), which makes it possible quantitatively to evaluate the suitability of this solution. The purpose of the operation must be formalized in it.

Thus, from this it is clear that methodologically operations research can be divided into the following steps: formulation of a mathematical model of the operation, selection of the criterion of the operation, determination of information necessary to study the model, and study of the mathematical problem to find the optimal solution.

General questions of operations research are considered in more detail in the following paragraphs of Chapter 1 and in Chapters 2 and 3.

Problems of operations research can be divided into two large groups.

1. Problems of the selection of optimal methods of using tehcnical devices (in the military they are called tactical problems. For brevity this term will also be retained in the future). A feature of these problems is that characteristics of technical devices are assumed to be given and known. It is necessary to select optimal methods of using them.

2. Problems of selecting optimal characteristics of technical devices (in the military they are called technical problems). These problems, as a rule, are much more complex than the first, as with a change in the characteristics of technical devices, optimal methods of using them can also change. Therefore, in solving a technical problem, it is frequently necessary to solve a tactical problem as well.

Problems of the selection of optimal methods of using technical devices are described in the basic literature on operations research [1-13]. The second problem is given much less attention, even though the possibilities of using methods of operations research for solving technical problems are much greater. Until the present, not one monograph has been published dealing with the solution of technical problems of operations research.

Let us discuss in more detail the content of technical problems, keeping in mind the aims of the study.

Such problems primarily include the problem of determining the suitability of developing a new technical device to replace an existing one, or more generally, the problem of determining suitable times for the development of new technical

devices. Let us point out that here we are speaking of the development of new, more improved technical devices having better technical characteristics than existing ones. Solution of this problem should help answer the first question, arising at the start of designing a new technical device: is it worth beginning work at all; perhaps it would be better to use previously developed technical equipment. Description of this problem is dealt with in Chapter 4.

The second technical problem of operations research is selecting optimal series of technical devices; i.e. selection of their basic characteristics. Any technical device performs various tasks. The narrower the range of these tasks, the better it can accomplish them and the more economically does it accomplish them. From this point of view, it is better to have many types of technical devices. But, on the other hand, creation of each new type requires expenditures for their development and additional expenditures for the operation of different types of devices. From this point of view it is better to have as few types of devices as possible. The existence of these two contradictory tendencies also necessitates determining optimal series. These problems are considered in Chapter 5.

The third technical problem of operations research is selecting the optimal combination of design parameters. A typical problem of this class is determining optimal reliability (durability). The higher the reliability of a technical device, the lower are operational expenses and the higher its productivity. However, increase of reliability requires additional expenditures, which also gives rise to the problem of selecting optimal reliability. Besides this problem, a large number of questions arises in designing a technical device which must be resolved, taking into account contradictory demands (increase of throughput with reduction of cost; increase of flight distance with increase of load-carrying capacity, etc.); therefore, the third group of technical problems of operations research is extremely broad. Several problems of this group are dealt with in Chapters 6 and 7.

The fourth group of technical problems are problems of comparative evaluation of the selection of the best of "competitive" technical devices, developed, for example, in a competition. As a rule, one technical device will exceed another in some parameters and will be inferior in others; therefore, selection of the better is quite complicated. Operations research methods also often make it

possible to solve problems belonging to this group. These problems are not considered in this book.

All problems of operations research (both tactical and technical) are now usually divided, according to content, into the following types: distribution problems, problems of stock control, problems of replacement, problems of mass service, controversial problems, ordering problems, search problems and problems of selection of optimal motion systems.

1. Distribution problems. There is a limited amount of resources, insufficient for performing all tasks in the best way. It is necessary to distribute them in a way to achieve the greatest effect. Such problems must be dealt with very frequently in solving technical problems of operations research. These problems include problems of optimal reserves, selection of reliability of parts, distribution of clearances between units of technical devices, etc.

2. Problems of stock control. Maintaining stock requires certain expenditures; lack of stock leads to expenditures connected, for example, with the loss of equipment time. It is necessary to determine the optimal amount of stock or, in the more common case, the order of replenishing it. Technical problems of operations research are also frequently problems of stock control. For example, the problem of selecting optimal volume of replacement parts, and many others.

3. Problems of replacement of equipment consist of determining the times and, more commonly, the order of replacing technical devices, taking into account their physical wear and obsolescence, based on demands for minimizing total expenditures. An attempt is now being made to create a special scientific discipline--the theory of restoration [26], dealing with study of the function of technical devices during their lifetime. Technical problems of operations research often use methods of solving problems of the replacement of equipment. In particular, these problems include, for example, determining optimal times for the development of new kinds of technical devices and determining optimal times for preventive replacement of elements, parts and units.

4. Problems of mass service investigate optimal methods of the use and characteristics of mass service systems, i.e. systems in which the time of servicing an order is a random value, and orders come at random moments of time. The function of such systems is studied by a special science--the theory of mass service, which is treated by numerous works [43, 36 and 44]. Problems of mass service are encountered in solving technical problems of operations research, as many technical devices (from telephone exchanges to repair shops) can be considered as mass service systems.

5. Controversial problems consider two sides, whose interests are contradictory, and study the question of the optimal strategies (i.e. methods of action) of these sides. Many controversial problems can be solved by methods of the theory of games [22, 35]. Situations considered in these problems are typical of the military and of competition. Technical problems of operations research rarely lead to controversial problems.

6. Ordering problems commonly consider selection of the order of service, in which a certain criterion (for example, time of service) is minized or maximized. One direction of these problems-- methods of power line planning [30]--is now being developed rapidly, but problems solved by these methods bear, as a rule, a tactical character. At the same time, methods of power line planning can also be applied to designing certain kinds of technical devices where operations are carried out in a certain order.

7. Search problems are formulated as follows. It is necessary to search for a certain object (defective cell, necessary book, defective article in a group of finished products, or a target in an enemy position). Resources are limited. A search procedure must be found which will maximize the probability of finding the object. Search problems basically bear a tactical character; however, they can also be encountered in designing certain technical devices, for example, those with equipment to search for failures. Search problems are now being formulated into a scientific discipline--the theory of search [23].

8. Problems of the selection of optimal motion systems are quite varied. The first problem of this class was that of the commercial traveler (to visit all given points with a minimum

of expenditures). Later they were widely developed in connection with the necessity of finding optimal conditions for the motion of aircraft, primarily rockets [25].

By character of the calculation of time, all problems of operations research can be divided into three groups. Static problems of operations research do not include study of the effect of change in time of the volume and character of solved problems and the possible improvement of technical devices on optimal solutions. Dynamic problems are the concern of the study of the dynamics of development. Kinematic problems study the character of development without taking into account the "inertia of the process," i.e. the expenditure of time to develop new types of technical devices and their production.

In conclusion, let us dwell on three methodological features of operations research.

The first is the use of a specific scientific method: selection of a criterion, construction of a mathematical model of the studied process, determination of necessary information and search for the optimal solution using mathematical programming methods.

The second feature is the systems approach. Its essence is the necessity of considering the fact that change in any part of the system can lead to change in the function of the entire system. Therefore, it is necessary to determine all important changes, and solving problems of operations research, definitely calculate these interconnections. This, of course, does not mean a need to overcomplicate operations research models by calculating unimportant connections.

The third feature is the necessity of using the achievements of different sciences: the theory of designing technical devices, economics, physics, mathematics, engineering psychology and many others.

1.2. THE HIERARCHY OF A PROBLEM OF OPERATIONS RESEARCH

The first question which arises in constructing a model of operations research is how broad the model should be.

In fact, all phenomena in life are connected with each other and a change in any characteristic of technical devices to some degree affects all surrounding phenomena. For example, in selecting the optimal parameter for a technical device it is necessary to consider the entire national economy of the country as a whole, as changes in parameters of this device will somehow affect all branches of the national economy.

However, solution of operations research problems in this way is impossible for primarily two reasons:

-- the incredible complexity of such a study,

-- the exceptionally weak effect of the studied parameter on general indexes of the national economy.

Therefore, from the total process of development of the national economy an individual part is separated, its model is considered and individual characteristics of technical devices are selected which are optimal in one sense or other of the word.

The narrower this part, the simpler it is to construct a model and the more strongly will parameters of technical devices affect indexes of the considered part of the process.

But, on the other hand, having narrowed the area of consideration down, we might fail to take into account important connections between the considered part of the process and its other parts and, as a result, commit gross errors.

Thus, the question arises of what rules can be followed in choosing the size of the part of the process of development of the national economy, a model of which is needed for solving a particular technical problem of operations research.

The entire national economy of the country can be represented as the interaction of a number of its branches.

The most general scheme of the national economy, making it possible to consider the most important laws, was described by Karl Marx. It specified separation of the economy into two large parts: production of the means of production and production of consumer items. This two-directional model makes it

possible to reach very important conclusions about necessary relations between these branches. To reveal laws existing within these subdivisions, more detailed models are also considered in economics--multi-branch models, primarily four-branched models: production of equipment and the means of labor, production of objects of labor, production of objects of consumption and creation of basic nonproductive stock [21].

A larger number of branches, reaching several dozen, is also considered in economics for more detailed studies.

Economic models at this stage of the hierarchy do not consider any specific characteristics of technical devices used in the branch, but generalized economic indexes of these branches are introduced into the calculation. Optimal sizes of these branches are determined as a result of the study.

They depend on appropriations assigned to development of this branch, the volume of tasks (operations) performed in this branch, etc.

USSR State Planning Committee competition is concerned with solving problems of this class.

Generalized economic indexes, which are operative in these models, depend on specific characteristics, technical devices used in the branch, and can be determined only in models at lower stages of the hierarchy.

The most important generalized characteristics of technical devices used in economic models are: service time of the technical device (k_i), productivity per unit of time, measured by corresponding cost (C_τ) and cost of the device itself (C_1), as well as generalized index $\frac{k_1 \cdot C_\tau}{C_1}$, i.e. the ratio of summary productivity during the existence of the technical device to its cost.

Thus, in inter-branch models of economics problems are distributed among branches; generalized economic indexes are used whose calculation requires data obtained from models occurring at lower stages of the hierarchy of models, which we shall discuss later.

After economic models of the national economy follow models connected with choice of the optimal quantitative ratio and optimal technical characteristics of various technical devices. They can be divided into three classes:

1) models which consider the interaction of various kinds of means, radically differing in function. For example, sea, river, railroad, air and motor transport (first class);

2) models which consider the interaction of the same kind of means (i.e. having no radical difference in function), but of different types. For example, sea vessels with different load-bearing capacity, airplanes with different flight distance, etc. (second class);

3) models which consider the interaction of means of the same kind and same type (third class).

These models are intended for solving problems of choosing optimal characteristics and number of technical devices.

Features of models of the first class:

-- the impossibility of completely replacing one kind of means by others and frequently the impossibility of one kind of means functioning without the others. For example, delivery of freight by sea, then by river, and finally by motor transport. As a result, an important part of these models is a large number of logical connections;

-- the large size of such models makes it impossible, as a rule, to consider all units of all types of means of each kind and, as a result, a generalized means (or a number of similar means) of each kind, selected optimally, is introduced into the consideration. For this reason, solving problems with such models requires information from models of following classes, in which optimal combinations of optimal means of each type must be selected;

-- models of this class, as a rule, are used to solve distribution problems. Optimal combination of different kinds of means are selected in them and operations accomplished by different kinds of means distributed.

Ministry planning agencies primarily encounter such models.

Features of models of the second class:

-- action of the same kind of means, but of different types, which do not perform the entire volume of operations, but only a part. Therefore, solving problems with these models requires information on the volume of operations performed by devices of a given kind, which is received from models of the first class;

-- in using models of this class it is assumed that devices of each type have optimal characteristics; therefore, their solution requires information obtained from the third class of models.

Models of this class, as a rule, are used to determine the basic characteristics and optimal number of types of devices of a given kind. Such problems are sometimes called problems of choosing optimal series (load-bearing capacity of ships, flight distance of airplanes, etc.). In these problems the volume of operations performed by devices of each type is also determined.

Models of the second class are used in developing specific technical devices usually in advanced scientific-research institutes.

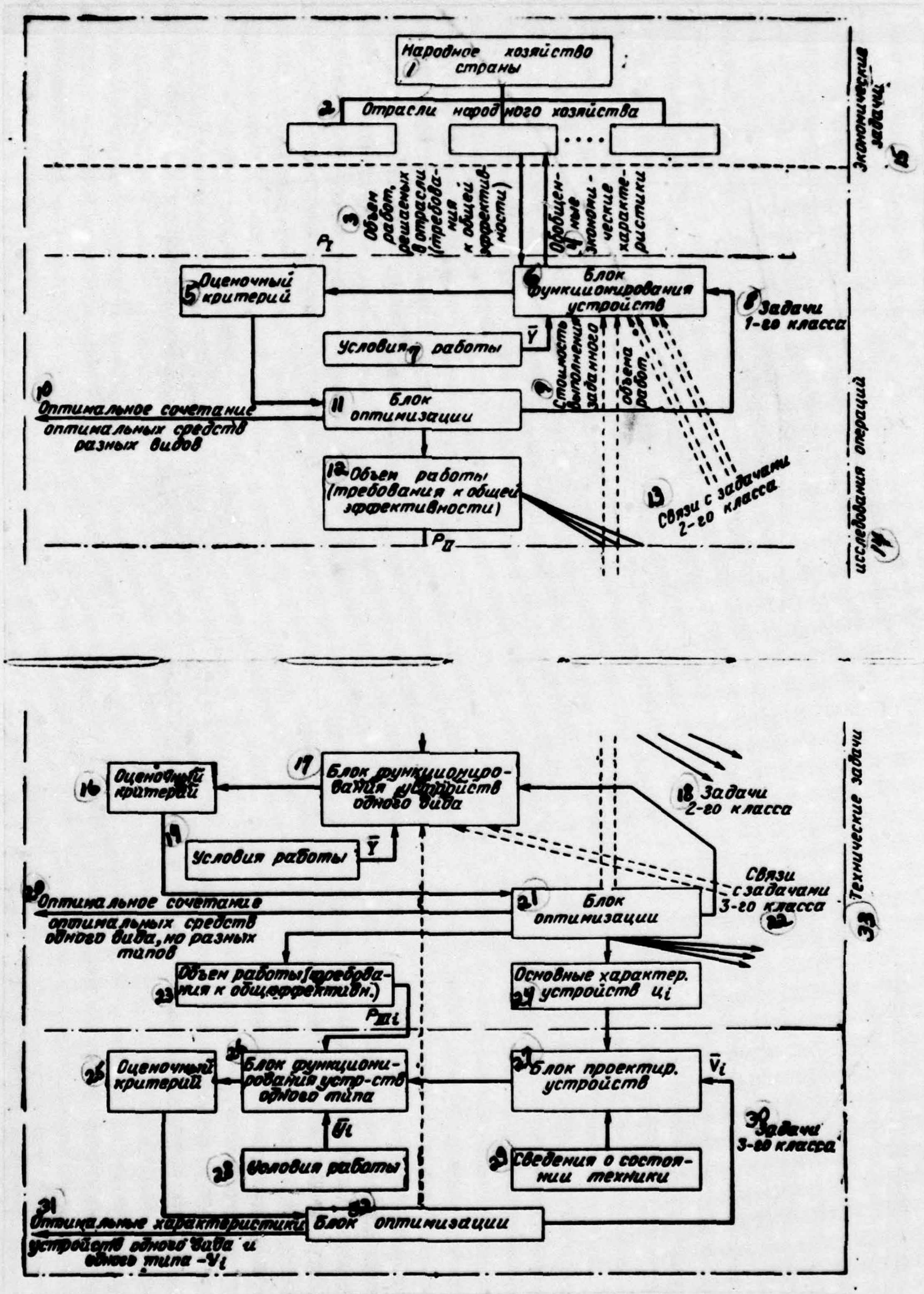
Features of models of the third class:

-- action of means of one type and one kind, which perform part of the total volume of operations. As a result, their use requires information on the volume of operations obtained from models of the second class.

Models of the third class are used to determine optimal characteristics of devices of each type with given basic characteristics (typical of certain types). Such problems are in essence generalized problems of optimal design.

These problems are solved directly in design bureaus and scientific-research institutes.

The general structural scheme of the hierarchy of problems and connections between them are shown in Fig. 1.2.1., where P_I, P_{II} and P_{III} indicate the



(Key on following page.)

Figure 1.2.1. Scheme of hierarchy of problems of optimization of technical devices.

Figure 1.2.1. Scheme of hierarchy of problems of optimization of technical devices.

1. National economy of country
2. Branches of national economy
3. Volume of operations performed in branch (demands for total effectiveness)
4. Generalized economic characteristics
5. Evaluation criterion
6. Device function block
7. Working conditions
8. Problems of 1st class
9. Cost of performing given volume of operations
10. Optimal oombination of optimal means of different kinds
11. Optimization block
12. Volume of operations (demands for total effectiveness)
13. Connections with problems of 2nd class
14. Operations research
15. Economic problems
16. Evaluation criterion
17. Block of function of one kind of devices
18. Problems of 2nd class
19. Working conditions
20. Optimal combination of optimal means of one kind, but different types
21. Optimization block
22. Connection with problems of the 3rd class
23. Volume of operations (demands for total effectiveness)
24. Basic characteristics of devices u_i
25. Evaluation criterion
26. Block of function of devices of one type
27. Device design block
28. Working conditions
29. Information on state of the art
30. Problems of 3rd class
31. Optimal characteristics of devices of one kind and one type
32. Optimization block
33. Technical problems

volume of operations performed by devices of a given branch, given kind and given type; ν_i — parameters of technical devices.

From the above and from the scheme we can see the connection between models of all classes and the necessity of simultaneously considering models of the entire hierarchy, which in practice is impossible.

It is possible, of course, to use the method of sequential approximations, giving input data in the first approximation by heuristic method, and then refining them. In a number of cases this is exactly how one must proceed.

The most convenient method would be to consider individual models, unconnected with each other, if this would ensure acceptable accuracy of the results.

As a number of practical examples of solving problems by this method led to poor results, it was better to estimate the value of deviations from accurate solutions due to errors in assumptions and initial data.

Let us consider an approximate estimate of the role of small deviations from accurate solutions of problems of optimization, presented by I. B. Pogozhev, with the following assumptions.

Accurate solution of each problem of optimization for estimating deviations can be considered as a solution of a corresponding problem of minimization of total expenditures $C(\bar{\xi})$ with given demands for total effectiveness of the devices. Therefore, true are the ratios

$$W(\bar{\xi}+\Delta\xi)=W(\bar{\xi})=\bar{W}, \tag{1}$$

$$C(\bar{\xi}+\Delta\xi)>C(\bar{\xi})=\bar{C}, \tag{2}$$

where $\bar{\xi}=\{\xi_1, \ldots, \xi_m\}$ — is the "accurate" and $\bar{\xi}+\Delta\xi=\{\xi_1+\Delta\xi_1, \ldots, \xi_m+\Delta\xi_m\}$ the "approximate" solution of the problem of optimization; $\Delta\xi_1, \ldots, \Delta\xi_m$ — deviations from optimal values of the considered characteristics; $W(\bar{\xi})$ — the index of effectiveness of technical devices; $\bar{W}$ — its given value. The quality of the approximate solution of the problem of optimization and the role of corresponding deviations $\Delta\xi_1, \ldots, \Delta\xi_m$ can be estimated by the relative change in total expenditures

$$\frac{\Delta C}{C}=\frac{C(\bar{\xi}+\Delta\xi)-C(\bar{\xi})}{C(\bar{\xi})}. \tag{3}$$

Functions $W(\bar{\xi})$ and $C(\bar{\xi})$ are twice continuously differentiated according to all arguments.

Deviations $\Delta\xi_1\ldots,\Delta\xi_m$ are small enough that in estimating the value of $\Delta C/C$ one need only calculate terms to the second order of smallness inclusively in relation to the value of deviations and to consider deviations only in characteristics whose values do not lie on the boundary of the permissible area of change of the latter.

Let $\xi_1\ldots,\xi_m$ be optimized expenditures for the considered technical devices, so that

$$C(\bar{\xi})=\sum_{k=1}^{m}\xi_k. \tag{4}$$

Let us assume that in the proximity of accurate solution $\bar{\xi}$ small relative changes in the assumed index of effectiveness $W(\bar{\xi})$ are approximately proportional to corresponding relative changes in expenditures $\Delta\xi_1\ldots,\Delta\xi_m$. Then in this proximity function $W(\bar{\xi})$ can be approximated by the expression

$$W(\bar{\xi})=W\prod_{k=1}^{m}\left(\frac{\xi_k}{\xi}\right)^{\beta_k}, \tag{5}$$

where $\beta_1,\ldots,\beta_k,\ldots,\beta_m$ — parameters.

Using in expanding expression (3) to a series ratios (4), (5) and condition (1), we obtain a simple formula

$$\frac{\Delta C}{C}=\frac{1}{2}\delta^2+O(\delta^3), \tag{6}$$

$$\delta^2=\sum_{k=1}^{m}a_k\delta_k^2, \tag{7}$$

where $a_1=\frac{\xi_1}{C},\ \ldots,\ a_m=\frac{\xi_m}{C}$ — relative optimal expenditures;

$\delta_1=\frac{\Delta\xi_1}{C},\ \ldots,\ \delta_m=\frac{\Delta\xi_m}{C}$ — relative deviations from optimal expenditures.

δ^2 is the "weighted mean" value of relative deviations and characterizes the accuracy of approximate solution of the problem of optimization.

Calculations show that if values $a_1 \ldots, a_m$ (or $\delta_1 \ldots, \delta_m$) do not differ sharply from each other, then in formula (7) averaging with "weights" a_k can be replaced by ordinary averaging, i.e.

$$\delta^2 = \frac{1}{m} \sum_{k=1}^{m} \delta_k^2. \tag{8}$$

The quality of the approximate solution can be estimated according to formulae (6) and (8) without knowing the values of relative optimal expenditures $a_1 \ldots, a_m$.

Let us consider a system of problems of optimization connected with each other so that solution of some problems will serve as initial data for solving others. Let us analyze the role of deviations from accurate solutions of problems of optimization in reverse order to that described above, i.e. we shall "move upward on the hierarchical ladder of problems."

Let us assume as the result of solving problems of optimization of characteristics of one kind and type of technical devices (third class) the costs of the latter are determined; these serve as initial data for problems of optimization of a group of technical devices of one kind (second class) and solutions of the latter determine average (for each group) costs of one device, which are used as initial data for problems of optimization of technical devices of different kinds (first class).

Generalizing this scheme, we obtain a system of problems of optimization for different classes in which solutions of problems of optimization of n-class give initial data of the costs of elements used as initial data in problems of optimization of $n - 1$ class.

We shall assume that solutions of problems of optimization obtained with inaccurate initial costs of technical devices are used as the basis for composing a plan of creating these devices in certain amounts, which is realized regardless of what costs will be obtained actually.

Let us assume that for each problem of optimization the assumptions formulated above are made, but different problems of optimization of one class do not contain general initial data and deviations for such problems can be considered as independent. With these assumptions in the considered system of problems

$$\frac{\Delta C_n}{C_n} = \frac{\delta^{2^{n+1}}}{2^{n+1}} \quad (n = 0, 1, 2, \ldots, N-1), \tag{9}$$

where δ characterizes the accuracy of solution of problems of N-class and is determined by formulae (7) or (8).

For example, if optimal characteristics of technical devices of one kind and one type are determined on the average with 45% accuracy ($\delta \cong 0{,}45$), then the relative change in total expenditures is about 10% in problems of the given class, approximately 0.5% in problems of optimization of groups of technical devices of one kind and only 0.001% in problems of optimization of technical devices of different kinds.

Ratio (9) shows that the role of small deviations from accurate solutions of particular problems of optimization is sharply reduced in transfer to more general problems.

Problems of the optimization of different classes can therefore be solved simultaneously.

This property is closely connected with the fact that optimization is conducted at each stage of the hierarchy, thanks to which connections between models occurring at these stages become weak. Otherwise, the indicated property could not exist.

Thus, in this case, if optimization is conducted at each stage of the hierarchy, models of each stage can be considered independently. Errors due to the introduction of inaccurate information can be estimated with the aid of equation (9).

1.3. FUNCTIONAL MODELS OF TECHNICAL DEVICES

In order to determine optimal characteristics of technical devices, it is necessary to construct models of their function. Two theoretically different models must be distinguished: models of the function of units and parts of a technical device inside it (internal models) and models of the function of a technical device as a whole in the second sphere (external model)[16]. The first model provides more detailed study of the function of the technical device, refining its characteristics, and it can be used in designing the device.

The second model helps determine optimal characteristics of a technical device and is basic in operations research. Construction of this model for the designer is, as a rule, more complex than the first; therefore, let us consider general aspects of modeling, concentrating primarily on functional models of technical devices in the second sphere.

A brilliant example of the construction of a mathematical model and its study to answer a specific question (the possibility of the development of capitalism in Russia under conditions of the destruction and impoverishment of the peasantry, leading at first glance to curtailment of the market) is given in the work of V. I. Lenin, "Po povodu tak nazyvayemogo voprosa o rynkakh" ("Regarding the so-called question of markets").[1]

In this work Lenin calls the mathematical model a scheme.

Before constructing the model, Lenin clearly defines the purpose of the study for which it is being created. Then he explains the essence and content of the basic processes occurring in society (in this case conversion of the natural economy of direct producers to a commodity economy and conversion of the commodity economy to a capatilistic). The next step in construction of the model (scheme) is explanation of the factors affecting the development and flow of these basic processes. The scheme (model), writes Lenin, should be compiled in such a way to show the effect of these processes and causes, their results.

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[1]V. I. Lenin. Works, 5th ed., Vol. 1,p. 67-123.

V. I. Lenin writes: "The following scheme is also compiled according to this plan: all aspects of the situation are abstracted, i.e. assumed to be invariable (for example, population, productivity of labor and much else), in order to analyze the effect on the market of *only* the indicates moments in the development of captialism."[1]

Having constructed a scheme (model) meeting the above indicated requirements, Lenin performed according to this scheme numerical calculations and after analysis of the data reached a conclusion about the possibility of the development of capitalism in Russia.

Thus, the process of construction of a mathematical model can be represented as consisting of the following steps:

1. Clear definition and formulation of the purpose of the study in order then to work out a model (scheme) meeting this purpose. It is difficult to construct a model whose study will answer a number of questions, and practically impossible to construct a model answering all questions, as in this case the model must approximate the original.

2. Explanation of the basic processes of the studied phenomenon, their essence, the factors causing these processes and determining their development.

3. Composition of a model (scheme) to show these basic processes and changes caused by them, in the basic index of the study. All outside circumstances (processes) are abstracted in order to analyze the effect on the final result of these basic processes. The latter significantly distinguishes compilation of a model (scheme) from study of the subject. In the first case we construct a model that will answer one specific question, in the second--we are attempting to study all aspects of the phenomenon.

However, in constructing a model it is necessary to consider all processes and interactions in order to distinguish the basic from the secondary (constructed for the purpose of a given study).

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[1]V. I. Lenin. Works. 5th ed., Vol. 1, page 87.

The following basic processes can occur in any model of the function of a technical device: obtaining information, control, motion, production and maintenance.

The first process consists of obtaining information needed for control. This is, for example, measuring the temperature of a furnace, calculating the number of parts made, computing the stock of blanks, etc.

Control consists of collecting information, processing it, making calculations to substantiate an optimal solution, making a decision, sending it to the executor and monitoring its performance.

The necessity of movement in space is connection with the nature of the production process and in a number of cases can be absent.

Production plays a basic role in modeling the function of technical devices and, as a rule, must be described quite fully.

All processes which are described above, as a rule, need maintenance: material-technical (supply of raw material); energy (supply of energy); repair (possibility of repair); transportation, including road construction, communications; facilities, etc. Consideration of all these processes is an extremely complex matter. Therefore, in construction of a specific model one must of necessity be limited to consideration of a minimum of maintenance processes, using only those with which interconnections are essential.

In modeling it is important to keep in mind one more thing. The majority of elementary processes or elementary phenomena, from which the entire process or phenomenon in question is composed, in its nature contains elements of randomness and their results are random. At first glance it can also seem that the general result will be random, and not conforming to a rule (necessary). This, however, is far from the case.

V. I. Lenin writes: "If value is an exchange ratio, then it is necessary to understand the difference between an individual, exchange ratio and a fixed ratio, between random and mass, between momentary and covering long periods of time. If it is so--and it is undoubtedly so--we shall inevitably rise from

random and individual to steady and mass, from value to cost."[1]

Thus, individual and momentary are typical of the random; mass, covering long periods of time — of the law-conforming.

The appearance of a non-random result in models of large phenomena, the elementary parts of which are characterized by random results, is called stochastic determinism [51].

The conditions of its manifestation are the following:

1. Large size and duration of the process, the measure of which can be the number of elementary phenomena N occurring in the considered process in the considered length of time. The coefficient of variation of the index of the entire process decreases approximately proportional to $\sqrt{N}$.

2. Lack of domination (superiority) of one random phenomenon over all the rest. Domination of one random process over the others leads to an increase in the coefficient of variation of the index of the entire process approximately proportional to $\sqrt{1+V^2}$, where V is the measure of domination, coefficient of variation of mathematical expectations of indexes of elementary phenomena.

3. Lack of correlations between the results of all elementary phenomena. For development of stochastic determinism it is sufficient that the number of non-zero coefficients of correlation be less than N^2, or that these coefficients approach zero, with N approaching infinity.

The phenomenon of stochastic determinism in many cases facilitates study of models of complex phenomena, making it possible to ignore elements of randomness.

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[1]V. I. Lenin, Works. 5th ed., Vol. 25, p. 47.

1.4. MODEL OF DESIGNING TECHNICAL DEVICES

When the methods of operations research are used to solve technical problems, a large number of variants of the technical devices must be designed in order to compare them later. Therefore, the use of ordinary methods of design in this case is difficult because of their great complexity. What is more, it is usually not necessary to develop units and parts in detail to a degree making it possible to manufacture them.

Therefore, approximate designing of technical devices can be automated (as a rule, using electronic computers).

The design block is a mathematical model of the process of designing technical devices, allowing use of given basic characteristics of this device and determining characteristics (ultimately reflecting the level of the development of science, technology and production) to calculate all parameters of the technical device, optimally designed, necessary for subsequent analysis.

Depending on the volume of information necessary for designing, the design block can be either a collection of a small number of very simple formula, or very complex algorithms.

In the usual case, the design block is a hierarchical ladder of design problems. For example, the design of electronic computers includes a hierarchical ladder, at the top of which is the design of the computer as a whole, distribution of functions and volume among basic mechanisms: arithmetical, memory, input and output devices, power system. This part of the model greatly resembles a model of the function of diversified technical devices from the preceeding paragraph.

Then follows a level resembling the level of problems of choosing optimum types, for example, selection of volume and other parameters of operative memory, intermediate memory and long-term memories.

The next levels are design of supports, blocks and cells. The same examples of calculation of optimization problems can be used as described in §1.2.

We must note that a number of problems of optimization can also be solved within the level, it is the easiest part of the problem. Design itself can arbitrarily be divided into three steps: selection of the theoretical base (scheme) of the technical device, performance of calculations to determine the type and sizes of parts and units, as well as the configuration of the technical device; these steps can be repeated both at different stages of design (pre-draft, draft and technical design steps), and within each of these steps.

The first step of design is the most complicated, as it is based on combinations of existing principles, ideas, schemes and units under the condition of observing certain logical connections between them.

What is quite easy for an experienced designer to perform is extremely difficult to describe by an algorithm, realized on a computer.

The construction of a model of the second step of design, as a rule, is simpler than the other steps because the overwhelming majority of stages in this processes are quite well formalized; there are mathematical dependences for calculating certain parameters.

The configuration of a technical device consists of the spatial placement of its units and parts while observing certain limitations (for example, in overall dimensions) and of trying to fulfill several conditions (for example, minimum distances between individual units, no intersections in connections between units, etc.). In this case the problem can amount to simplified geometry by representing units and parts in the form of simple outline shells. In more complex cases the area in a field of receptors must be represented using a double code. The third step is simpler than the first, but more complex than the second.

The basic difficulty which is encountered in constructing a design block is that there is an enormous number of possible design plans for a technical device. Thus, in work [28], a calculation is made that the number of different combinations of only the basic units of a guided missile is over 3,000,000.

Calculation of all possible variants, even if calculation of each one is quite simple, presents great technical difficulties.

Ways of overcoming these difficulties consist of using logical rules, using heuristic methods and using results of the theory of dimensions.

Careful analysis of the basic principles of the function of a technical device and experience of its design in a number of cases helps establish certain logical design rules, which can be expressed in the impossibility of combining certain types of units or, vice versa, in the advisability of certain combinations, the equivalence of certain combinations, etc.

The use of some of these rules can greatly reduce the number of considered variants or merely determine a rational order of their analysis when the most interesting variants will be considered first.

Heuristic methods consist of the formalization of design experience, which can be carried out both in the form of logical rules of the type indicated above (but in this case their strict justifications are absent), and in introduction of any kind of empirical formulae, order of actions, estimates according to empirical rules, etc.

Definite results can also be given by the use of the theory of dimensions, making it possible by conversion from dimension values to dimensionless parameters to greatly reduce the number of different values and thereby significantly facilitate the problem of design.

The question of design automation is now in the initial stage of development if we are speaking of design for the purpose of using the results to produce technical devices.

However, if we are speaking of the design of different variants for comparative evaluation in operations research models, then important simplifications can be introduced and simple algorithms obtained. It must be kept in mind that in simplifying a model, in no case should the effect on individual processes of those parameters being studied be excluded.

1.5. CRITERIA AND LIMITATIONS

A criterion (purposeful function, index of operation) must meet the following basic requirements: it must be representative, critical to the studied parameters, as simple as possible, include results of all basic essential processes of the considered operation, and correctly take into account the stochistic nature of the process.

The representative quality of a criterion means evaluation of the basic (and not secondary) problem of the operation. Despite the obviousness of this requirement, errors arise in solving practical problems because the basic aim of the operation is not explained and, therefore, the criterion is formulated incorectly.

Being critical to the studied parameters means that significant changes in the numerical value of a criterion relate to comparatively small changes in the studied parameters. In a number of cases, a high critical level facilitates mathematical studies.

It is extremely desirable that the criterion be individual (and, therefore, include results of all other processes), as in the case of two or more criteria the study is significantly complicated, and in some cases becomes simply impossible.

As a rule, random processes are considered in operations research. Usually a mathematical expectation of the criterion is considered and its minimum or maximum is found. However, in those cases when its dispersion is great, its reliable intervals must also be considered, in order to estimate to what degree the selected solution is guaranteed.

The basic postulate of operations research is as follows. The optimal solution is that which ensures performance of a given task with the minimum of material expenditures (direct formulation). The reverse formulation is also possible. The optimal solution is that which ensures maximum effectiveness (performing a maximum of tasks) with fixed material expenditures.

From that, the general form of the criterion in direct formulation of a problem of operations research will be a mathematical expectation of material expenditures (which are usually expressed as cost) with a given effectiveness (in this case it is a limitation), i.e.

$$\min K = \min M\left(\sum_{i=1}^{l} C_i\right); \qquad (1)$$
$$Э \geqslant Э_{доп}.$$

In the case of the reverse formulation of the problem, the general form of the criterion will be effectiveness $Э$ with given material expenditures, which are a limitation, i.e.

$$\max Э$$
$$M\left(\sum_{i=1}^{l} C_i\right) < M_{доп}. \qquad (2)$$

We must note that it is not always required to use criteria in the most general form. In fact, if material expenditures are expressed in similar products, then there is no need to convert to their cost, and we need only determine their number, weight, volume, etc.

The criterion of effectiveness can be not only production cost (this is its common form), but also the number of production articles, if similar products are being manufactured, the amount of effective time that a technical device operates, etc.

The common form of the criterion must be used when its simpler forms cannot be used.

The selection of a criterion is quite a complex problem, in a number of cases requiring independent studies and, in particular, analysis of models of higher classes.

In the majority of technical problems of operations research both formulations of the problem (direct and reverse) are equivalent, i.e. they have the same solution. It was possible to show that in carrying out certain conditions, optimum solutions, found with any formulation of problems of operations research, will be the same. These conditions include the following:

-- increased effectiveness, connected with improved quality of the technical device according to any parameter, leads to an increase or at least maintenance of its value;

-- there is at least one parameter of the technical device, the improvement of which will increase the value of the device.

This significantly facilitates the practical solution of problems of operations research, making it possible to choose the formulation which is more convenient for the study. It must be stated that this property of problems of operations research is a reflection of the principle of duality, which occurs in many mathematical problems.

Besides limitations connected with the common form of problems of operations research, other kinds of limitations are also encountered in solving these problems. They can be limitations of weight and over-all dimensions of technical devices, limitations in permissible technical characteristics (for example, in strength of materials, permissible temperatures and mechanical loads of radio-electronic elements), limitations on the number of personnel, etc.

These limitations are primarily connected with the fact that only part of the model is considered and broken connections with other parts are replaced by limitations.

Depending on the mathematical methods used in finding optimal solutions, limitations complicate or simplify solution of the problem.

1.6. CHARACTERISTICS OF TECHNICAL DEVICES

Characteristics of technical devices are as diverse as the technical devices themselves. However, with all their diversity, they can be divided into four large groups: 1) productivity, 2) reliability, 3) cost and 4) weights and overall dimensions of technical devices. Let us consider them briefly, keeping in mind their use in solving technical problems of operations research.

P r o d u c t i v i t y of t e c h n i c a l d e v i c e s can be characterized by the number of operations which the technical device performs per unit of time. A characteristic of productivity can also be the value which results from the work of a technical device per unit of time. The most important characteristics of productivity are limitations on possible area of use of the technical device. Thus, for turning lathes it is the limitation of overall dimensions of work pieces, rate of delivery, engine power; for electronic computers it is limitations on volume of memory, etc.

R e l i a b i l i t y o f t e c h n i c a l d e v i c e s can be characterized by the following indexes for a case of non-recoverable technical devices:

-- probability of failure-proof operation of the device during time t

$$P(t)=\int_{t}^{\infty} f(t)\,dt, \tag{1}$$

where $f(t)$ — density of distribution of time of failure-proof operation of the device;

-- average service time of the device

$$t_{\text{cp}}=\int_{0}^{\infty} t f(t)\,dt; \tag{2}$$

-- intensity of failures of devices

$$\lambda(t)=\frac{f(t)}{P(t)}=-\frac{1}{P(t)}\frac{dP(t)}{dt}, \tag{3}$$

i.e. conditional probability of failure is close to moment of time t.

Let us note that $f(t)$, $P(t)$ and $\lambda(t)$ — interconnected values, and knowing one of them, we can find all the others.

If the time of failure-proof operation of the device is distributed according to exponential rule, i.e.

$$f(t)=\lambda \exp(-\lambda t), \tag{4}$$

then

$$P(t) = \exp(-\lambda t). \tag{5}$$

$$t_{\text{ср}} = \frac{1}{\lambda}. \tag{6}$$

$$\lambda = \text{const}. \tag{7}$$

For recoverable products, besides the above indicated index, others must also be considered which are connected with repair of the articles, in particular operational readiness

$$k_{\text{г}} = \frac{t^{*}_{\text{ср}}}{t^{*}_{\text{ср}} + T_{\text{в}}}. \tag{8}$$

where $T_{\text{в}}$ — average time of finding and eliminating a failure; $t^{*}_{\text{ср}}$ — average time of operation before failure (mean cycles between failures).

A more complete index of the technical device, also taking into account lost time due to preventive operations, is the utilization factor

$$k_{\text{и}} = \frac{t^{*}_{\text{ср}}}{t^{*}_{\text{ср}} + T_{\text{в}} + T_{\text{проф}}}. \tag{9}$$

where $T_{\text{проф}}$ — average time of preventive treatment per failure, occurring in the considered length of time.

C o s t o f t e c h n i c a l d e v i c e s is one of the most important characteristics, largely determining suitable areas and even the theoretical possibility of using particular devices. It is characterized by the following values:

1) cost of development of a given kind of technical device $C_{\text{р}}$, including the cost of conducting scientific-research operations, designing, producing and testing prototypes of technical devices;

2) cost of production of one model of the technical device $C_{\text{п}}$;

3) cost of operation of one model of the technical device per year $C_{\text{э}}$ or of its storage $C_{\text{х}}$ (in the future we shall use one symbol $C_{\text{х}}$).

The cost of developing a technical device is usually expressed in relation to the cost of production of one (the first) model C_0

$$k_p = \frac{C_p}{C_0}. \tag{10}$$

For different technical devices coefficient k_p changes within quite wide limits. Evidently k_p is greater, the more new solutions, principles and ideas introduced in creating the new technical device or in the final analysis--the greater the increase in effectiveness of the device Э in comparison with existing devices Э_0 achieved in development of the new

$$k_p = F\left(\frac{\text{Э}}{\text{Э}_0}\right). \tag{11}$$

Analysis of statistical data shows that this function in a number of cases can be approximated by the following dependence:

$$k_p = k_{p0} \exp\left(\frac{\text{Э}}{\text{Э}_0}\right). \tag{12}$$

The cost of production of one model of the technical device to a significant degree depends on the volume and time of production. A dependence of the following kind is usually used in operations research:

$$C_{nN} = C_0 N^{\mu}, \tag{13}$$

where C_{nN} — cost of production of N devices, beginning with the first; C_0 — cost of production of the first device; μ — exponent, less than one, and in a number of cases close to 0.7.

The formula shows that with growth in production volume, the cost of each individual specimen is reduced.

The cost of the first model C_0 to a significant degree is determined by the features of the technical device and its accurate determination can be an extremely complex problem, solvable by methods of economics. However, in a number of cases, analysis of statistical material can produce comparatively simple formulae, connecting C_0 with the basic technical characteristics of different kinds of devices. Let us present as an example several such formulae which are

very convenient for use in problems of operations research.

First of all it is possible to determine the cost of production of a technical device, analogous to an existing device whose cost is known, by a comparison of weights Q

$$C_1 = C_2 \left(\frac{Q_1}{Q_2}\right)^a, \tag{14}$$

where a for technical devices of different kinds fluctuates between 0.7-0.4. Let us point out that here we are speaking of devices which are realized on similar principles and in which increase of size (weight) increases productivity. If new principles must be realized to reduce weight with a given productivity of the device, then formula (14) cannot be used.

There are general dependences connecting the cost of a part (unit) with the accuracy of their production δ and with reliability P. They have the following form:

$$C_o = A + \frac{B}{\delta^h}, \tag{15}$$

where A, B and h — coeffieients; δ — manufacturing tolerance;

$$C_o = \frac{A}{(1 - P)^a}, \tag{16}$$

where A and a — coefficients.

It is important to point out connection of C_0 with the coefficient of unification $k_{ун}$ (ratio of the total number of parts to the number of unified parts):

$$C_o \simeq \frac{A}{k_{ун}}. \tag{17}$$

For each type of technical devices, having statistical data and using the method of least squares, it is possible to obtain approximate dependences of the following kind:

$$C_o = \sum_{i=1}^{N} a_i k_i \tag{18}$$

or

$$C_o = \prod_{i=1}^{N} \xi_i^{b_i}, \tag{19}$$

where ξ_i — characteristics of technical devices; a_i and b_i — test coefficients.

In particular, for turning lathes

$$C_o \cong k_T D_y^{1.9}, \tag{20}$$

where D_y — greatest diameter of mounted article; k_T — numerical coefficient.

For press equipment

$$C_o \cong k_{\text{п}} G^{0.43}, \tag{21}$$

where G — force of press.

For trucks

$$C_o = (a + bQ)v, \tag{22}$$

where Q — load-bearing capacity; v — maximum speed loaded; a and b — coefficients.

An analogous approximation formula can also be used for determining the cost of helicopters using coefficients a and b.

The cost of airplanes with turboprop engines is approximately connected with its basic parameters by the following method:

$$C_o \cong aP_x + b\frac{P_x}{Q} + cv, \tag{23}$$

where P_x — thrust of the engine; Q — weight of the airplane; v — cruising speed; a, b, c — coefficients.

The cost of electronic computers can be represented by the following approximation function of their basic characteristics:

$$C_э = k_э(БП)^{0.3}m^{0.6}, \qquad (24)$$

where $Б$ — response speed; $П$ — volume of operative memory; m — machine discharge.

The cost of operation of a technical device per unit of time is determined by expenditures for energy (electricity, fuel), routine and major repoairs, and for personnel.

In operations research the ratio of expenditures for operation per year to the cost of an individual model:

$$k_э = \frac{C_э}{C_о} = \frac{C_{эн}k_{эн} + C_{рм}k_{рм}\varphi(t) + C_л k_л}{C_о}, \qquad (25)$$

where $C_{эн}, C_{рм}, C_л$ — cost per unit of fuel or electricity, cost of repair, and average salary of personnel; $k_{эн}, k_{рм}, k_л$ — quantitative index of power demand, number of repairs, and personnel; $\phi(t)$ — function indicating increase of expenditures for operation as technical device wears out.

Analysis of test data shows that this function, as a rule, is closely approximated by a dependence of the following kind:

$$\varphi(t) = \exp(\mu t) - 1, \qquad (26)$$

where μ — test coefficient.

If the technical device is not operated, but is kept in storage, then certain expenditures $C_х$ per unit of time are also required:

$$k_х = \frac{C_х}{C_о} = \frac{C_{скл}}{C_о} + \frac{1}{C_о}\sum_{i=1}^{m}\frac{C_{срi}}{a_i} + \frac{C_{об}N_р}{C_о}, \qquad (27)$$

where $C_{срi}$ — the cost of units of the technical device; a_i — period during which they are useless; $C_{об}$ — cost of personnel (technical, warehouse, guards); $N_р$ — number of units of personnel needed for one technical device; $C_{скл}$ — cost

of warehousing needed for one technical device.

Thus, usually the cost expended for N technical devices in time T, if they are not decommissioned, can be determined according to the following formula:

$$C_N = C_0\left[\frac{k_p}{N_n}N + N^u + \beta N\int_0^T k_0\,dt + (1-\beta)N\int_0^T k_x\,dt\right] \approx \approx C_0\left[\frac{k_p}{N_n}N + N^u + N\int_0^T k^*_x\,dt\right]. \qquad (28)$$

where β — proportion of operated articles; N_n — total volume of production.

This cost is a complex nonlinear function of the number of technical devices and time.

Expenditures made at different times are not equivalent to each other. The later they are made, the less they mean, and vice versa, the earlier they are made, the more weight they have. To calculate this in economics expenditures are reduced to a single moment of time using formula

$$C_{t=0} = \frac{C_{t=t'}}{a^{t'}}, \qquad (29)$$

where $a = 1 + \alpha$; α — coefficient of effectiveness of capital expenditures, the value of which is approximately 0.1.

Characteristics of weight and overall dimensions of technical devices are quite simple. They are extremely important for movable technical devices, especially those mounted in aircraft, and play a lesser role for stationary technical devices.

Characteristics of technical devices can change very strongly in time (technical progress leads to the possibility of significant improvement of these characteristics). The question of predicting characteristics of technical devices is dealt with in the following chapter.

2. PREDICTING THE CHARACTERISTICS OF TECHNICAL DEVICES

2.1. GENERAL ASPECTS OF PREDICTION

To solve any dynamic problem of operations research it is necessary to predict as a function of time a series of input data:

1) possible characteristics of technical devices;
2) demands for technical devices (volume of problems solved by these devices);
3) operating conditions (external) of technical devices.

We shall discuss in detail only the first question, as demands for technical devices are determined in models of the development of the national economy and its branches, as are operating conditions of technical devices (for example, the development of economics in northern or southern regions, etc.).

The prediction of scientific-technical development is scientifically based information on the future of science and technology. Scientific prediction is acquiring special urgency at present in connection with the regularly accelerating course of development of science and technology. Prediction must be based on the use of specific trends and actual laws of scientific-technical development. Some general laws of this process will be shown later. In order to use the results of prediction in models of operations research, these results must be presented in the form of specific numerical values with estimates of the accuracy of the prediction.

In predicting the performance of technical devices we must distinguish two possible types: determining ξ and derivative ξ_a.

We shall call determining characteristics those characteristics of technical devices, the analysis of whose development goes beyond the limits of the studied model. Such characteristics are, for example, specific strength of materials in considering the development of technical devices of a specific type, energy characteristics of fuels in studying the development of a narrow class of engines,

reliability, weight, overall dimensions of elements of communications devices in studying the development of electronic computer technology, etc.

Derivative characteristics we shall call those which are dependent on determining characterstics and are selected in each specific case in the studied model, taking into account economic feasibility.

For example, maximum flight range of transport airplanes is selected not only on the basis of determining characteristics, which limit technical possibilities of creating an airplane with a certain range, but also the demands for airplanes with a particular flight range, which will evidently never exceed 20,000 kg, as well as on the economic feasibility of choosing an airplane with a particular flight range.

If the trend in the development of determining characteristics is continuous growth (decrease), true, as is shown later, at different rates at different stages of development, then derivative characteristics increase (decrease) in accordance with the increase (decrease) of determining characteristics until demands are satisfied. Later these characteristics increase (decrease), remain unchanged, or decrease (increase) in accordance with changes in demands for these characteristics and their economic feasibility.

Determining characteristics can be calculated only by prediction methods, derivative characteristics by optimization according to one of the criteria of economic feasibility, using the predictions of determining characteristics, a design model of the technical device and a functional model of the technical device. A diagram of the determination of these determining characteristics is shown in Fig. 2.1.1. The essence of the problem is a search for the set of derivative characteristics which would ensure maximum (minimum) of the evaluation criterion. To solve this problem it is necessary to predict determining characteristics, the volume of tasks performed by the technical devices and working conditions of this device. Let us note that internal optimization can be conducted in both the design model of a technical device and in its functional model (for example, if some type of engine ensures creation of a technical device which exceeds other devices in all indices, then the technical device must be used with this engine).

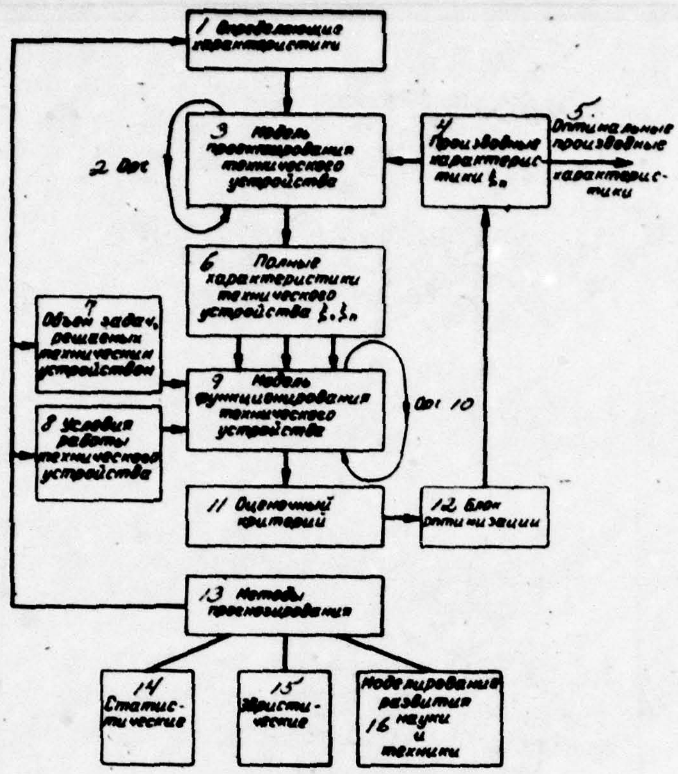

Figure 2.1.1. Diagram of the determination of derivative characteristics.

1. Determining characteristics
2. Unit vector
3. Model of design of technical device
4. Derivative characteristics ξ_n
5. Optimal derivative characteristics
6. Complete characteristics of technical device ξ, ξ_n
7. Volume of tasks solved by technical device
8. Operating conditions of technical device
9. Model of the function of technical device
10. Unit vector
11. Evaluation criterion
12. Optimization block
13. Methods of prediction
14. Statistical
15. Heuristic
16. Modeling the development of science and technology

Let us consider change in the determining characteristics of technical devices in several examples. Figures 2.1.2-2.1.5 show change in time in weight strength of materials used by man; caloricity of fuels used by man; speed of travel of man over long distances, and finally, response time of computers relative to their weight.

Analysis of these graphs reveals general laws inherent in the development of all these characteristics.

First of all, the process of change in determining characteristics is an alternation of periods of evolutionary and revolutionary development. The first are distinguished by slow change in characteristics, achieved by improving the technology of production, methods of design, etc. The second are distinguished by rapid change in characteristics, achieved by using new physical principles, technical ideas, etc. ..."Life and development in nature," says V. I. Lenin, "include both slow evolution and rapid leaps, intervals of gradualness."[1]

This regularity makes it quite difficult to predict the determining characteristics of technical devices, as the process of change in these characteristics usually cannot be described by a continuous function. Therefore, one of possible approximations must be used.

If we are speaking of a large period of time, in which many leaps are possible, then with a certain degree of approximation change in determining characteristics can be approximated by continuous dependence.

This is also the case in short-term predictions, when it can be expected that no jump will occur in the considered period. Finally, difficulties in predicting do not arise if it is known where the leap can lead. We must note that from the moment an idea appears, even if it is now being implemented, until there is a sufficient number of technical devices in which this idea is used, there is a long period of time, sometimes lasting decades.

Prediction for not too great and not too small an interval of time is extremely difficult; in this case one possible approximation will be a step function.

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[1]V. I. Lenin. Works, 5th ed., Vol. 20, p. 66.

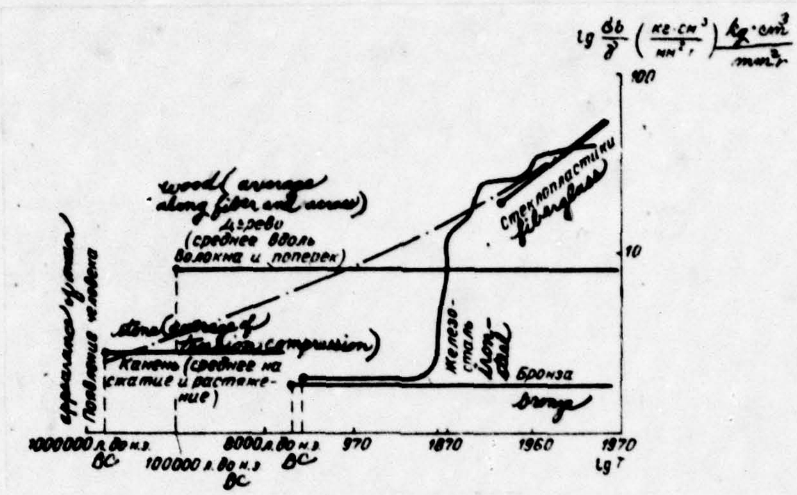


Figure 2.1.2. Change in time in ultimate strength in relation to specific weight of materials used by man

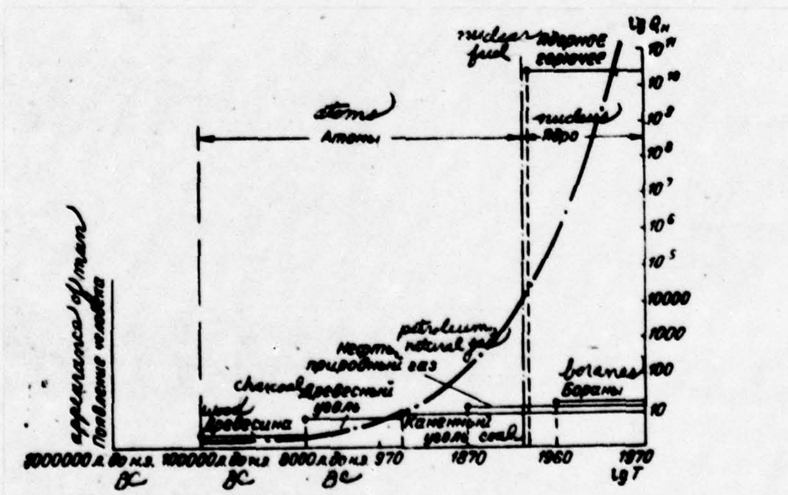


Figure 2.1.3. Change in time in caloricy of fuels used by man, per unit of weight, burned in air (except nuclear fuels)

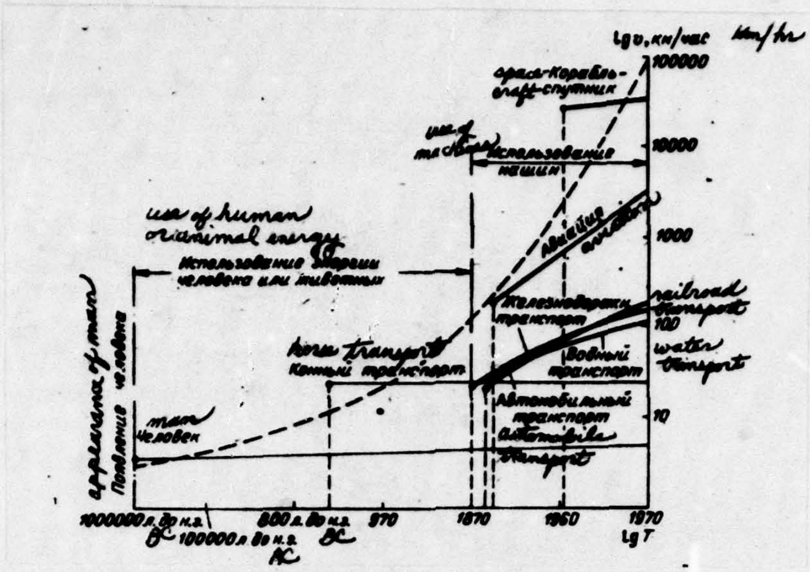


Figure 2.1.4. Change in time in speed of human transportation over long distances

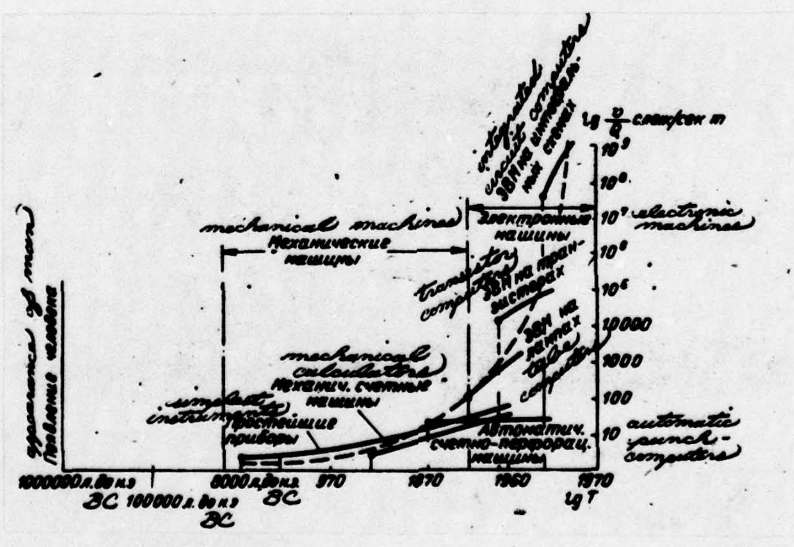


Figure 2.1.5. Change in time in response time of computers in relation to their weight

Secondly, the process of change in determining characteristics has a tendency toward increased frequency of leaps (i.e. toward reduction of the intervals of time between them) and increase of their value (if we are speaking of increasing characteristics).

Here, first of all, we must distinguish leaps of different categories.

The most significant are leaps connected with the use of new principles (for example, the use of atomic energy, space travel, the introduction of electronic computers, etc.).

The next category is connected with the use of new solutions within limits of the same principle (for example, the use of different fuels for combustion in air, conversion from steam engines to diesels, etc.).

Finally, the technology of production or design also improves in leaps whose values are immeasurably smaller than those preceeding.

Therefore, we shall only consider *the first two groups of leaps.* The frequency of these leaps (distance between them) regularly decreases; it is perfectly clear that strict conformity to law is here to a certain degree obscured by unknown random elements. On the average, the time between leaps decreases exponentially, which is satisfactorily described in the following form:

$$\Delta\tau = 10^{5,5-i}, \tag{1}$$

where i — number of the leap.

As regards the size of leaps, this is a much more complicated matter. There are leaps (the majority) when the introduction of a new principle or new solution within limits of an old principle immediately leads to a sudden increase in some determining characteristic. However, there are also leaps when this characteristic is first lower than that achieved in using an existing principle, but then with time in the process of evolution overtakes those which the existing principle makes possible (this is easiest to see in the change in weight strength of materials).

Thus, leaps can be either positive or negative (but in the latter case with a large time derivative). The value itself of positive leaps can either increase or decrease with time; decrease of leaps is observed in proportion to the exhaustion of possibilities of new solutions within limits of the existing principle. On the average, within limits of an existing principle, the size of leaps, determined by new solutions, corresponds to a dependence of the type $A[1+\mathrm{th}(a+bt)]$, i.e. early in the mastery of a principle leaps increase and then they decrease toward zero. As regards leaps due to the introduction of new principles, they have a clearly pronounced tendency to increase (a principle can be exhausted, but the general possibilities of development are inexhaustable); sizes of leaps increase approximately the same, following an exponential dependence. It can be noted that conversion to new principles makes it possible to change individual determining characteristics 10^4-10^{10} times, the use of new solutions within limits of the same principle — 10-10^4 times.

Methods of prediction can be divided into two large groups: direct and indirect prediction. In the first case, those characteristics which which are necessary for subsequent analysis are directly predicted. In the second case, characteristics are predicted which are necessary for calculation of the required characteristic, but the latter is determined by calculation.

Prediction itself can be carried out by four methods: by modeling processes of the development of science and technology, by the statistical method (i.e. analysis of statistical material used as a basis for mathematical dependences, which are then used for extrapolation), by the heuristic method (i.e. in the end by the use of opinions of experts) and by combined methods.

Below are described the essence of the first three methods. In conclusion we must note that prediction is one of the most complex questions, in which at the present state of science we are never guaranteed against serious errors.

2.2. MODELING PROCESSES OF THE DEVELOPMENT OF SCIENCE AND TECHNOLOGY

Modeling the development of science and technology is a complex problem. However, certain results can be obtained by analyzing the most common laws of development.

Let us consider the process of reproduction on an expanded scale (it doesn't matter of what: materials, technical ideas, etc.) under conditions of limited resources. In this case it is sometimes assumed that

$$\frac{d\xi}{dt} = a\left(1 - \frac{\xi}{k}\right)\xi, \tag{1}$$

i.e. the rate of growth increases in proportion to the amount created and decreases in proportion to the exhaustion of resources.

Here ξ — measure of that created; a and k — coefficients.

Integrating (1), we obtain

$$\frac{1}{\xi} - \frac{1}{k} = \left(\frac{1}{\xi_0} - \frac{1}{k}\right)\exp[-a(t - t_0)]. \tag{2}$$

Assuming for computing origin $t = t_0 = 0$, then

$$\frac{1}{\xi} - \frac{1}{k} = \left(\frac{1}{\xi_0} - \frac{1}{k}\right)\exp(-at),$$
$$\xi = \frac{k}{1 + \left(\frac{k}{\xi_0} - 1\right)\exp(-at)}. \tag{3}$$

Having denoted $b = \left(\frac{k}{\xi_0} - 1\right)$, we obtain an equation of a logistic curve in canonical form

$$\xi = \frac{k}{1 + b\exp(-at)}. \tag{4}$$

We note that this equation can also be described in the following form:

$$\xi = A[1 + \mathrm{th}(a_1 + b_1 t)]. \tag{5}$$

Actually, substituting the value of th, we obtain

$$\xi = A\left[1 + \frac{\exp(a_1 + b_1 t) - \exp(-a_1 - b_1 t)}{\exp(a_1 + b_1 t) + \exp(-a_1 - b_1 t)}\right] = \frac{2A}{1 + \exp(-2a_1 - 2b_1 t)},$$

and designating $2A = k$, $\exp(-2a_1) = b$, $2b_1 = a$, we obtain equation (4).

If $b \gg 1$, i.e. $\frac{k}{\xi_0} > 2$ or $\frac{\xi_0}{k} < \frac{1}{2}$, which physically stands for a low degree of use of general resources, we obtain exponential dependence

$$\xi \approx \frac{k}{b} \exp(at) \approx \xi_0 \exp(at). \tag{6}$$

Finally, with low at, which physically stands for small deviation from the initial state, we obtain linear dependence

$$\xi \approx \xi_0 (1 + at) = \xi_0 + \xi_0 at. \tag{7}$$

Thus, the process, analogous to the process of expanded reproduction under conditions of limited resources, is described by a logistic curve (function of a hyperbolic tangent); with unlimited resources — by an exponential curve; and with small changes — by a straight line.

Let us consider now a more common model. Let there be many principles, each of which is developed according to the dependences described above. The rate of search for a new principle is proportional to accumulated potential ξ, but the effect of each new principle grows in proportion to accumulated potential.

Then with unlimited resources for general development and limited resources for each principle, we can record that

$$\xi(t) = \max_i \left\{ \xi_0 \exp(k_1 i) \operatorname{th} \left[\left(t - \sum_{j=0}^{i} \exp(-k_2 j \right) \right] \right\}, \tag{8}$$

where i — the number of the principle; $a,\ \xi_0,\ k_1,\ k_2$ — positive coefficients.

If the value of a is great, then within the limit will occur step function

$$\xi = a_0 + \sum_{j=1}^{n} a_j \mu (t - t_j), \tag{9}$$

where

$$u(t - t_j) = 1 \quad \text{при} \quad t - t_j > 0;$$
$$u(t - t_j) = 0 \quad \text{при} \quad t - t_j < 0.$$

Dependences (5), (6), (8) and (9) can be used as the basis for statistical methods.

2.3. PREDICTION OF DETERMINING CHARACTERISTICS OF TECHNICAL DEVICES USING THE METHOD OF LEAST SQUARES

As indicated above, determining characteristics of technical devices can be predicted with the use of both continuous and step functions. First let us consider the first case. The following can be used as continuous functions (Fig. 2.3.1), taking into account what was stated in § 2.2:

linear $$\xi = a + bt; \tag{1}$$

exponential $$\xi = \xi_0 \exp(bt); \tag{2}$$

hyperbolic $$\xi = A[1 + \mathrm{th}(a + bt)]. \tag{3}$$

Thus, let there be information on change in certain determining characteristics ξ in time, represented in the form of a set of N points ξ_1, t_1, ξ_2, t_2, ..., ξ_N, t_N . In the case of linear function [54]

$$\xi = a + bx, \tag{4}$$

where

$$x = t - \frac{1}{N}\sum_{i=1}^{N} t_i; \tag{5}$$

$$a = \frac{1}{N}\sum_{i=1}^{N} \xi_i; \tag{6}$$

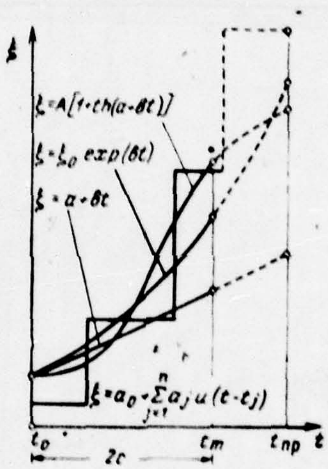
$$b = \frac{\sum_{i=1}^{N} \xi_i x_i}{\sum_{i=1}^{N} x_i^2}. \tag{7}$$

Errors ξ, determined by equation (4),

$$\sigma_\xi = \frac{\sigma}{\sqrt{N}} \sqrt{\frac{1}{\sum_{i=1}^{N} x_i^2}\left[N x^2 + \sum_{i=1}^{N} x_i^2\right]}. \tag{8}$$

where σ — root mean square deviation of ξ_i from the general trend of development, which can be determined approximately by the formula

$$\sigma < \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\xi_i - a - bx_i)^2}. \tag{9}$$



[$Bt = W$; $c = 5$]

Figure 2.3.1. Kinds of functions used for prediction.

As deviations of ξ_i from the general trend are connected with a great many factors, there is every basis for assuming that they are distributed according to normal law. In this case solution by the method of least squares coincides with solution according to the principle of maximum probability.

In the case of an exponential function, the following approximation method can be used:

$$\xi = \xi_0 \exp(bt). \tag{10}$$

From which

$$\ln \xi = \ln \xi_0 + bt. \tag{11}$$

We designate

$$\ln \xi = y, \tag{12}$$

$$\ln \xi_0 = a. \tag{13}$$

Let us now move from t to a new variagle x on the condition that $\sum_{i=1}^{N} x_i = 0$, where x_i — ordinates of points at our disposal. It is evident that

$$x = t - \frac{1}{N} \sum_{i=1}^{N} t_i. \tag{14}$$

Then

$$y = a + bx. \tag{15}$$

According to existing information we can calculate y_i and x_i and determine a and b by the method of least squares for linear function:

$$a = \frac{\sum_{i=1}^{N} y_i}{N}, \quad b = \frac{\sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2}. \tag{16}$$

Errors in initial information σ can be calculated approximately on the basis of deviations of test points from calculated points

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (y_i - a - bx_i)^2}{N-1}}. \tag{17}$$

Then by analogy with (8) we can record

$$\sigma_y = \sigma \sqrt{\frac{1}{N \sum_{i=1}^{N} x_i^2} \left[N x^2 + \sum_{i=1}^{N} x_i^2 \right]}. \tag{18}$$

from which

$$\sigma_y = \frac{\sigma}{\sqrt{N}} \sqrt{1 + \frac{N x^2}{\sum_{i=1}^{N} x_i^2}}. \tag{19}$$

If points x_i are distributed evenly over a given period $2c$, then it is not difficult to show that with a sufficiently large number

$$\sum_{i=1}^{N} x_i = \frac{Nc^2}{3}, \tag{20}$$

whence

$$\sigma_y = \frac{\sigma}{\sqrt{N}} \sqrt{1 + 3\left(\frac{x}{c}\right)^2}. \tag{21}$$

From the formula it is evident that error is minimum in the middle of the considered period and then increases, first relatively slowly, and at the edge — in proportion to x.

Let us consider the question of the distribution of values of ξ, calculated using formula (10).

With large N distribution $\sum_{i=1}^{N} y_i$ and $\sum_{i=1}^{N} x_i y_i$ can be considered normal; therefore, the distribution of y, calculated with the aid of formula (15), will also be normal. From that, the distribution of ξ, calculated on the basis of (11), will be logarithmically normal.

If analysis is conducted not in natural logarithms, but in common, which is easier, the mathematical expectation can be described in the following form:

$$M(\xi) = 10^{M(y)} \exp\left(\frac{\sigma_y^2}{2 \cdot 0.4342^2}\right). \tag{22}$$

i.e. a certain correction occurs, the value of which depends on the value of σ_y.

Mean square deviation of ξ is

$$\sigma(\xi) = M(\xi)\sqrt{\exp\left(\frac{\sigma_y^2}{0.4343^2}\right) - 1}. \tag{23}$$

An accurate method of calculating ξ_0 and b will be the following. It is necessary to minimize functional

$$D = \sum_{i=1}^{N} [\xi_i - \xi_0 \exp(bt_i)]^2 \tag{24}$$

by appropriate selection of ξ_0 and b.

Let us note that possible boundaries of change of ξ_0 are from ξ_{min} to ξ_{max}. Then the following search procedure can be used.

With fixed ξ_0, the value of b is sought by one of the methods of regular search, minimizing D. Then an analogous procedure is repeated in relation to b with the found ξ_0, then again in relation to ξ_0, etc., until the minimum function of these two variables is found.

The formula for linear function can be used as an approximation formula for evaluating the accuracy of the dependence obtained in this way.

In the case of hyperbolic dependence, an accurate method of solving the problem will be the following. It is necessary to minimize by means of selecting A, a and b the functional

$$D = \sum_{i=1}^{N} \{\xi_i - A[1 + \operatorname{th}(a + bt_i)]\}^2. \tag{25}$$

One of the existing search methods can be used for this.

The following can be recommended as an approximation method.

From argument t let us proceed to x according to formula (14). From variable ξ we proceed to variable y:

$$y = \operatorname{Ar\,th}\left(\frac{\xi}{A} - 1\right). \tag{26}$$

Later, according to existing information (points ξ_i and t_i) and the results of analyzing the process of development, we shall determine the value of A (this is the limit to which technical characteristics aim) and then we calculate

$$a = \frac{\sum_{i=1}^{N} y_i}{N},$$

$$b = \frac{\sum_{i=1}^{N} x_i y_i}{\Sigma x_i^2}, \quad (27)$$

as well as dispersion D using formula (25). The accuracy of this dependence is evaluated in first approximation with the aid of the formula for linear dependence.

Let us consider prediction using a step function. In this case the equation for a generalized technical characteristic resembles (2.2.9).[1]

To determine this function essentially we must know two functions:

$$a_j = f_a(j). \quad (28)$$

$$\delta t_j = f_t(j). \quad (29)$$

keeping in mind that

$$t_j = \sum_{j=1}^{n} \delta t_j.$$

Usually a_j — increasing function j (i.e. the size of leaps grows in proportion to technical progress), and δt_j — decreasing function j (i.e. leaps increase in frequency in proportion to technical progress). One possible variant of determining these functions in the presence of initial information at N points (ξ_i, t_i) will be the following.

The value of n (number of leaps) is determined by analyzing the history of development of a given kind of technology for the period 2 c.

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[1]Here and later figures in references to formulae indicate: first — number of the chapter, second — number of the paragraph, third — number of the formula.

We assume

$$\delta\tau_j = \tau \exp(-k_\tau j). \tag{30}$$

Then

$$2c = \tau \sum_{j=1}^{n} \exp(k_\tau j). \tag{31}$$

Taking into account that on the right is recorded the sum of geometric progression with denominator exp ($-k_\tau$), we obtain

$$\tau = 2c\,\frac{1-\exp(k_\tau)}{1-\exp(-k_\tau n)}. \tag{32}$$

Now, setting k_τ, we can calculate τ, δt_j and t_j, and then at each interval from t_{j-1} to t_j – calculate

$$M(a_j) = \frac{1}{n_j}\sum_{z_{j1}}^{z_{j2}} \xi_i - \sum_{k=1}^{j-1} M(a_k). \tag{33}$$

where z_{j1} and z_{j2} – numbers of the first and last points falling within a given interval, as well as

$$\sigma = \frac{1}{N-1}\sum_{j=1}^{n}\sum_{z_{j1}}^{z_{j2}} [\xi_i - M(a_j)]^2. \tag{34}$$

Optimal will be that k_τ at which σ is minimal. This algorithm is easily realized on any computer. The method of random search can also be used to solve this problem.

Before we considered statistical prediction using functions obtained by analyzing general models of development. An advantage of such an approach is the introduction of certain *a priori* information, which is especially important with limited statistical material.

Besides the described methods for prediction, it is also possible to use the method of constructing regression equations [40], which in the presence of a large

amount of statistical material can give very good results, and what is more important, can make it possible to estimate the accuracy of the results. Classical regression analysis is constructed in Taylor series expansion of the sought function

$$f(t)=\sum_{n=0}^{N} a_n x^n. \tag{35}$$

With the same success, Fourier series expansion or expansion ratios of Legendre, Laguerre and Hermite polynomials can be used for this purpose.

The use of Chebyshev polynomials is extremely effective in a number of cases

$$f(t)=\sum_{n=0}^{N} \cos(n \arccos x). \tag{36}$$

Also possible are more complex approaches to the problem of prediction, based on the use of methods developed in the theory of automatic control.

2.4. HEURISTIC PREDICTION

The general principle of heuristic prediction consists of using the opinions of experts to determine prospective development of technical devices.

The process of heuristic prediction can be arbitrarily divided into a number of stages.

1. Prediction of the development of natural sciences, which can be used to create or improve theoretically new technical devices.

2. Acquainting technological experts with this prediction; they, by conducting analysis, calculations and certain design studies, and possibly also by using statistical prediction methods, should be able to indicate possible characteristics of a given technical device, which can be achieved in a certain period of time.

3. Statistical analysis of results, obtained by these experts independently of one another. It is usually necessary to obtain the distribution function of expected characteristics, but sometimes only determination of mathematical expec-

tation and dispersion of this characteristic is enough.

If these estimates differ sharply from each other, the experts can evaluate the results jointly. Then a second independent evaluation of possible characteristics is conducted by the experts and the results subjected to subsequent statistical analysis.

The expected costs of technical devices with characteristics found as indicated above can be determined by querying economics experts.

It is very important to organize the questionnaire of experts so that certain conditions are observed:

1) statement of the questions in questionnaires should exclude the possibility of different interpretations. The experience of conduct expert estimates shows that this condition is far from always observed;

2) registering estimates should take a minimum of the expert's time. Experts are extremely busy people. This must be taken into consideration in compiling the questionnaire. The best form for the questionnaire will be that which presents a specific question and several numbers, one or several of which the expert is to encircle;

3) secrecy should be observed in conducting expert evaluations (it should not be known what evaluation the expert is giving). Violation of this condition will inevitably lead to "overcautious evaluations," due not to the opinion of the experts on a particular question, but to attendant circumstances (the expert's job; his relations with other experts, etc.).

Various kinds of expert evaluations are possible. First of all, we can obtain an estimate of the most probable value of the sought characteristic (incidentally, experts very rarely make a distinction between mathematical expectation of a certain value and its most probable value). We shall call such an evaluation ξ_i (i-expert). Then we can analyze the estimates of several (N) experts and according to known formulae obtain a mathematical expectation and dispersion of this value.

If the number of experts is quite large (which, considering the relatively small number of highly qualified experts and their very busy schedules, is almost never possible to achieve), then a histogram of values of ξ_i can be constructed, and then it could be approximated by some known distribution. In particular, Charlier distribution can be used to approximate distribution [54].

As an example, Fig. 2.4.1. shows test distribution of ξ, obtained from the results of 72 expert evaluations, and its approximation to normal distribution.

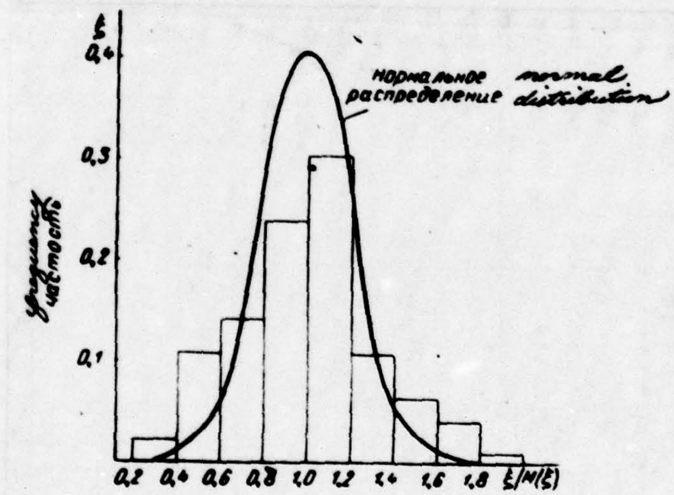

Figure 2.4.1. Test distribution of expert evaluations of ξ.

From the illustration it is evident that this distribution agrees closely with normal distribution and in the absence of an adequate number of expert evaluations, normal distribution can be used in first approximation (as is done in other cases).

A second kind of expert evaluations can be obtaining possible limits of a particular characteristic. We shall call the maximum value of a characteristic given by i-expert b_i, and the minimum $- a_i$. In this case, experts supply the researcher more information than in the preceeding; experience shows that experts give this form of information most willingly.

Many methods of information processing are possible. The difference between them is in the use of one or another a priori law of distribution of the characteristic between the extreme estimates given by the experts.

A particular case of β-distribution can be used:

$$f(\xi_i)=\frac{12}{(b_i-a_i)^4}(\xi_i-a_i)(b_i-\xi_i)^2 \qquad (1)$$

with $a_i < \xi_i < b_i$; $f(\xi_i) = 0$ in all other cases. For it

$$M(\xi_i) = \frac{2b_i + 3a_i}{5}, \tag{2}$$

$$D(\xi_i) = 0{,}04\,(b_i - a_i)^2. \tag{3}$$

In the presence of estimates (N) of experts

$$M(\xi) = \frac{1}{N}\sum_{i=1}^{N} M(\xi_i), \tag{4}$$

$$D(\xi) = \frac{1}{N}\sum_{i=1}^{N} D(\xi_i), \tag{5}$$

$$f(\xi) = \frac{1}{N}\sum_{i=1}^{N} f(\xi_i). \tag{6}$$

Some authors suggest introducing "weighted factors of experts," but because of the complete arbitrariness of determining these factors, the impossibility of preserving the secrecy of evaluations with their use and, finally, because of the inevitable resentments which would arise, the use of such factors cannot be recommended.

Instead of distribution (1) we can use logarithmically normal distribution of the kind

$$f(\xi_i) = \frac{\sqrt{2}}{(b_i - a_i)\sqrt{\pi}} \exp\{-2\,[\ln(\xi_i - a_i) - \ln(b_i - \xi_i) + 1]^2\}. \tag{7}$$

For which

$$M(\xi_i) = \frac{1{,}4a_i + b_i}{2{,}4}, \tag{8}$$

$$D(\xi_i) = 0{,}04\,(b_i - a_i)^2. \qquad (9)$$

Different authors recommend different distributions, in particular, gamma-distribution, Weibull, Erlang distribution, etc.

However, no one has proven the legitimicy of using a particular distribution and it is best to use the simplest distribution - equally probable distribution.

In this case

$$f(\xi_i) = \frac{1}{b_i - a_i} \text{ при } a_i < \xi_i < b_i, \qquad (10)$$

$f(\xi_i) = 0$ in all other cases;

$$M(\xi_i) = \frac{a_i + b_i}{2}, \qquad (11)$$

$$D(\xi_i) = \frac{1}{12}(b_i - a_i)^2. \qquad (12)$$

Later, using formulae (4), (5) and (6), with many evaluations, we can determine $M(\xi)$, $D(\xi)$ and $f(\xi)$, but the latter can be approximated using one of the known distributions.

Finally, experts might give three estimates: maximum value of the expected characteristic $\xi_{макс}$, minimum $\xi_{мин}$ and most probable ξ_{mod}.

In this case the following variety of β-distribution can be used:

$$f_i(\xi) = \frac{\frac{1}{\xi_{mod}}\left(\frac{1}{\xi_{mod}} + 1\right)}{(\xi_{макс} - \xi_{мин})\left(\frac{1}{\xi_{mod}} - 1\right)} \times$$
$$\times (\xi - \xi_{мин})(\xi_{макс} - \xi)^{\frac{1}{\xi_{mod}} - 1}. \qquad (13)$$

$f_i(\xi)$ computed, and then

$$f(\xi) = \frac{1}{N}\sum_{i=1}^{N} f_i(\xi) \qquad (14)$$

and, on the basis of this distribution $M(\xi)$ and $D(\xi)$ determined.

This method is much more complicated than the preceeding one. If we take into account the great "adjustment" of experts to two estimates, the preceeding method is preferable.

In conclusion, let us note that the most reliable method of prediction is a combination, including statistical prediction using functions obtained from general models of development and control of its results by the heuristic method. In those cases when the results of prediction by these two methods are close, statistical prediction can be used (the result of heuristic prediction can be used as an additional point). In those cases when the results of these methods differ significantly, statistical prediction cannot be used (we encounter a leap here). Trends in development must be carefully analyzed and, possibly, heuristic prediction repeated, the results of which can be taken as a basis.

Example 2.4.1. Known is the change in time t_i of the power-weight ratio of domestic automobile gasoline engines ξ_i (Table 2.4.1, columns 1, 2, 3 for 17 engines (N = 17)). Known are the results of the first interrogation of experts predicting this characteristic for 1980, according to which ξ = 1.2 kg/l·s. We must determine the expected power-weight ratio ξ_i for automobile engines in 1980 and the degree of reliability of the prediction.

Table 2.4.1

engine model Модель двигателя	ξ_i, кг/л.с. kg/hp	t_i	y_i	x_i	$x_i y_i$	x_i^2	$(y_i - a - bx_i)^2$
1	2	3	4	5	6	7	8
ГАЗ-А	4,09	1932	0,6117	—24	—14,6808	576	0,00790
ГАЗ-М1	3,64	1936	0,5611	—20	—11,2220	400	0,00949
ЗИЛ-101А	4,27	1936	0,6304	—20	—12,6080	400	0,00079
ГАЗ-М20	4,07	1946	0,6096	—10	— 6,0960	100	0,00317
ЗИЛ-110	2,88	1946	0,4594	—10	— 4,5940	100	0,00882
МЗМА-402	3,94	1956	0,5955	0	0	0	0,02173
ГАЗ-М21Б	3,23	1956	0,5092	0	0	0	0,00373
МЗМА-407	3,14	1958	0,4969	2	0,9938	4	0,00487
ЗМЗ-21Д1	1,79	1959	0,2529	3	0,7587	9	0,02676
ЗМЗ-13	1,23	1959	0,0899	3	0,2697	9	0,10670
ЗИЛ-111	1,90	1960	0,2788	4	11,1152	16	0,01618
МеМЗ-966	3,54	1962	0,5490	6	3,2940	36	0,02690
МЗМА-40721	3,14	1964	0,4969	8	3,9752	64	0,01769
МЗМА-408	2,92	1965	0,4654	9	4,1886	81	0,01254
МеМЗ-966А	3,20	1966	0,5051	10	5,0510	100	0,02631
МЗМА-412	1,76	1968	0,2455	12	2,9460	144	0,0584
ГАЗ-41	1,82	1969	0,2601	13	3,3813	169	0,0262
Σ	—	33238	7,6174	—	23,2273	2208	0,3010

First of all, using formulae (2.3.12) and (2.3.14), we sequentially calculate $y_i, x_i, x_i y_i, x_i^2$ (see Table 2.41, columns 4-7). Then we calculate a and b according to formulaes (2.3.16):

$$a = \frac{7,6174}{17} = 0,4481; \quad b = -\frac{23,2273}{2208} = -0,01052.$$

Using formula (2.3.17), after preliminary calculations in column 8 we calculate σ:

$$\sigma = \sqrt{\frac{0,3010}{17-1}} = 0,1371.$$

To determine expected value of ξ in the required year on the basis of formulae (2.3.12), (2.3.15) and (2.3.14), we write:

$$\xi = \exp\left[a + b\left(t - \frac{1}{N}\sum_i t_i\right)\right],$$

from which for 1980

$$\xi = \exp\left[0,4481 - 0,01052\left(1980 - \frac{33238}{17}\right)\right] = 1,596 \text{ kg/hp.}$$

Using formula (2.3.21) and taking into account that $c = \frac{t_{i\max} - t_{i\min}}{2}$ we calculate mean square error of the prediction

$$\sigma_y = \frac{0,1371}{\sqrt{17}} \sqrt{1 + 3\left(\frac{1980 - \frac{33238}{17}}{\frac{1969 - 1932}{2}}\right)^2} = 0,300 \text{ kg/hp.}$$

As the results of statistical prediction differ relatively little from heuristic prediction (less than σ_y), then we can assume that no leaps are foreseen in the considered period of time. Then both methods of prediction can be used and calculations repeated, having included the results of heuristic prediction (t_i = 1980; ξ_i = 1.2) as an additional point (N = 18).

As the order of calculations in this case is the same as in the preceeding one, we give only the results of calculation:

$$\xi = 1,556 \text{ kg/hp}, \quad \sigma_y = 0,221 \text{ kg/hp},$$

i.e. the mathematical expectation of the predicted value changes weakly, but accuracy increases significantly (σ_y is reduced almost 1.5 times).

3. MATHEMATICAL PROGRAMMING (OPTIMIZATION)

3.1. The Object of Mathematical Programming Classification of Methods

After a model of the operation is constructed, criteria are selected, limitations are determined and input information is obtained, which is necessary to find the optimal solution (plan, program). Solution of this problem is also the object of mathematical programming, which has noting to do with the formulation of programs for electronic computers.

We shall give a general statement of problems of mathematical programming in relation to operations research.

A model of an operation can usually be represented in the following form:

$$K = \Phi(x_1, x_2, \ldots, x_l;\ t;\ u_1, u_2, \ldots, u_k); \tag{1}$$

$$x_i \in x_{i\,\text{доп}} \quad \text{where} \quad i = 1, 2, \ldots, l; \tag{2}$$

$$u_k \in u_{k\,\text{доп}} \quad \text{where} \quad k = 1, 2, \ldots, K; \tag{3}$$

$$\begin{gathered} g_l(x_1, x_2, \ldots, x_l;\ x_1^{(1)}, x_2^{(1)}, \ldots, x_l^{(1)}; \ldots; \\ x_1^{(m)}, x_2^{(m)}, \ldots, x_l^{(m)};\ t;\ u_1, u_2, \ldots, u_K = 0 \end{gathered} \tag{4}$$

$$\text{where } l = 1, 2, \ldots, L;$$

$$\begin{gathered} \varphi_p(x_{10}, x_{20}, \ldots, x_{l0};\ x_{10}^{(1)}, x_{20}^{(1)}, \ldots, x_{l0}^{(1)}; \ldots; x_{10}^{(m)}, \\ x_{20}^{(m)}, \ldots, x_{l0}^{(m)};\ t_0;\ u_{10}, u_{20}, \ldots, u_{k_0} = 0 \end{gathered} \tag{5}$$

$$\text{where } p = 1, 2, \ldots, P; \ldots$$

$$\begin{gathered} \psi_q(x_{1\,\text{кон}}, x_{2\,\text{кон}}, \ldots, x_{l\,\text{кон}};\ x_{1\,\text{кон}}^{(1)}, x_{2\,\text{кон}}^{(1)}, \ldots, x_{l\,\text{кон}}^{(1)}; \ldots \\ \ldots; x_{1\,\text{кон}}^{(m)}, x_{2\,\text{кон}}^{(m)}, \ldots, x_{l\,\text{кон}}^{(m)}, t_{\text{кон}};\ u_{1\,\text{кон}}, u_{2\,\text{кон}}, \ldots, u_{K\,\text{кон}}) = 0 \\ \text{with } q = 1, 2, \ldots, Q, \end{gathered} \tag{6}$$

where x_i — coordinates of the system (dependent variables; t — independent variable; u_k — controls; symbols $x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)}$ designate first, second,...,m-derivatives according to the independent variable, the index "доп" — possible, "кон" — finite; $\in$ -- inclusion.

Here and later we are talking about so-called phase coordinates, which are a generalization of the concept of geometric coordinates. Phase coordinates are the minimal number of parameters which can be used to characterize the condition of the system in question. The condition of the system can be represented in the form of a point with these coordinates in a certain arbitrary phase space.

Equation (1) is the calculating formula of the criterion, conditions (2) and (3) are limitations on coordinates and controls, equations (4) are connections between coordinates, controls and the independent variable, equations (5) and (6) are initial and final conditions.

System (1)-(6) can be written in a briefer form, introducing vector designations:

$$K = \Phi(\bar{X}, t, \bar{U}); \quad \bar{X} \in \bar{X}_{доп}; \quad \bar{U} \in \bar{U}_{доп};$$
$$\bar{g}(\bar{X}, \bar{X}^{(1)}, \ldots, \bar{X}^{(m)}, t, \bar{U}) = 0;$$
$$\bar{\varphi}(\bar{X}_0, \bar{X}_0^{(1)}, \ldots, \bar{X}_0^{(m)}, t_0, \bar{U}_0) = 0;$$
$$\bar{\psi}(\bar{X}_{кон}, \bar{X}_{кон}^{(1)}, \ldots, \bar{X}_{кон}^{(1)}, t_{кон}, \bar{U}_{кон}) = 0. \tag{7}$$

Those controls (program, plan of actions) must be found which will maximize (or minimize) the criteria.

By the control (program) is understood a certain plan of actions, which can be continuous, multi-step or one-step. In the first case, the control (program) is a function of an independent variable (most often time). In a particular case the problem amounts to finding the parameters determining this function. In the second case, a set of parameters (numbers) determining the control at each step is sought. Finally, in the third case, only one step is considered and a set of parameters (numbers) characterizing the control is sought.

The kind of mathematical programming depends on the existence and kind of limitations, the number of steps in the operation and the form of the model, although the same problem can be solved by various methods.

Methods of mathematical programming can be divided into two groups: analytical and numerical. To the first group belong differential and variational calculus, L. S. Pontryagin's maximization principle and the methods of V. F. Krotov based on sufficient conditions. The second group includes dynamic, linear and nonlinear programming, methods of regular and random search. The use of analytical methods requires that the calculating formula of the criterion, limitations and connections between coordinates, controls and the independent variable, as well as initial and final conditions, be represented in the form of functions, which must be differentiated at least once and have a finite number of discontinuity points.

The use of classical methods (differential and variational calculus) also demands a lack of limitations. If the control is a set of numbers, then methods of differential calculus can be used; if the control is a function (or functions of independent variables), then calculus of variations methods can be used for the problem. In using numerical methods it is necessary to know the possible area of change of controls, and the narrower this field (i.e. the more limitations), the more effective is the use of numerical methods.

If there are limitations and the control is a function of independent variables, but the model is a set of analytical dependences, then we can use Pontryagin's maximization principle and Krotov's methods based on sufficient conditions.

Dynamic programming is used to study multi-step processes. Nominally it can be used in any cases, and the possibilities of electronic computers serve as an important limitation of its use. The discrete maximization principle also has analogous properties as it is an extension of Pontryagin's maximization principle to discrete processes.

If the criterion is a linear function of controls and limitations are a set of linear inequalities (equations) and the process is single-step, it is a classical problem of linear programming. In a number of cases, multi-step problems can also be reduced to a problem of linear programming.

If the criterion and limitations are nonlinear functions of controls, and the process is one-step, we come to the problem of nonlinear programming, for which no general methods of solution have yet been developed, despite the availability of numerous methods used for various particular cases.

Methods of sampling (regular or random) can be used to solve any one-step problem of mathematical programming. The possibilities of these methods are limited by laborious calculations.

The solution of mathematical programming problems is much more difficult if we must deal with random functions or values. But this is exactly typical of operations research problems. However, the development of methods of solving these problems is far from complete. Methods of linear programming relating to a given case (stochastic programming) and sampling methods (sampling in the presence of "noise") are being developed intensively.

Much attention has recently been given to heuristic programming, which is a unique bionic direction in the theory of optimization. Here we are no longer speaking of strict methods of solution. But at the same time, heuristic programming makes it possible to find a solution in those cases when other methods are useless.

Below we briefly describe the essence of various methods of mathematical programming and give their comparative evaluation.

In conclusion, let us dwell on the basic questions which must be faced in using particular methods of mathematical programming.

In using analytical methods, it is much easier to find a solution satisfying the necessary conditions of an extremum and much more complicated to prove their adequacy. Therefore, in practice, the fulfillment of necessary conditions is frequently limited and their adequacy not proven.

With the use of numerical methods, the principal question is global sampling, i.e. the greatest (or least) maximum (or minimum) and not a local (particular) extremum. There is always the danger of "falling into the trap" — the local extremum which must not be taken into account.

3.2. Finding the Greatest (Least) Values of Functions by Methods of Differential Calculus

To find the extremum (maximum or minimum) of a function with one variable $\Phi(u)$ it is necessary to find roots $\tilde{u}$ of equation

$$\frac{d\Phi(\tilde{u})}{du}=0. \tag{1}$$

If $\frac{d^2\Phi(\tilde{u})}{du^2}>0,$, then there is min.

If $\frac{d^2\Phi(\tilde{u})}{du^2}<0,$, then there is max.

If $\frac{d^2\Phi(\tilde{u})}{du^2}=0,$, then with $\frac{d^3\Phi(\tilde{u})}{du^3}\neq 0$ at point $\tilde{u}$ there is neither max nor min, and in general, if the derivative of an odd series is equal to zero, then there is neither max nor min, the sign of the derivative of an even series determines the presence of max or min.

For finding the greatest and least values, which the function takes in a given interval $u_0\leqslant u\leqslant u_{\text{кон}}$, , it is necessary to calculate all max and min in this interval, to calculate the value of the function at the ends of the interval, at discontinuity points of the function and at discontinuity points of its derivative. On the basis of a comparison of these values, the greatest or least value of the function is also determined.

In the case of a function with two variables $\Phi(u_1, u_2)$, a system of equations is solved:

$$\frac{\partial\Phi(u_1,u_2)}{\partial u_1}=0;\quad \frac{\partial\Phi(u_1,u_2)}{\partial u_2}=0. \tag{2}$$

The obtained systems of solutions $(\tilde{u}_1, \tilde{u}_2)$; ... are analyzed by substitution in $\frac{\partial^2\Phi(u_1,u_2)}{\partial u_1^2}$; $\frac{\partial^2\Phi(u_1,u_2)}{\partial u_2^2}$ and $\frac{\partial^2\Phi(u_1,u_2)}{\partial u_1\partial u_2}$ and by calculation of the following expression:

$$\Delta=\begin{vmatrix} \frac{\partial^2\Phi(u_1,u_2)}{\partial u_1^2} & \frac{\partial^2\Phi(u_1,u_2)}{\partial u_1\partial u_2} \\ \frac{\partial^2\Phi(u_1,u_2)}{\partial u_1\partial u_2} & \frac{\partial^2\Phi(u_1,u_2)}{\partial u_2^2} \end{vmatrix}. \tag{3}$$

If $\Delta>0$ and $\frac{\partial^2\Phi(u_1,u_2)}{\partial u_1^2}<0,$ then there is max; if $\Delta>0$ and $\frac{\partial^2\Phi(u_1,u_2)}{\partial u_1^2}>0,$ then there is min; if $\Delta<0,$, then there is neither max nor min.

If $\Delta=0,$ then analytical studies are difficult and it is necessary to calculate $\Phi(u_1, u_2)$ with values close to $(\tilde{u}_1, \tilde{u}_2)$, directly verifying the presence or absence of max and min.

In the case of a function with many variables $\Phi(u_1, u_2, \ldots, u_K)$ it is necessary to calculate roots of a system of equations

$$\frac{\partial\Phi(u_1, u_2, \ldots, u_K)}{\partial u_1} = 0; \quad \ldots; \frac{\partial\Phi(u_1, u_2, \ldots, u_K)}{\partial u_K} = 0 \tag{4}$$

and to analyze the function with values $|\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_K|$ with a direct substitution, as analytical conditions of max and min are complex.

The greatest or least values of $\Phi(u_1, u_2, \ldots, u_K)$ are found in a fixed area by the same means as in the case of a function with one variable; however, with a larger number of variables this problem becomes practically unsolvable and linear or nonlinear programming methods must be used.

From this it can be seen that the necessary conditions for the use of differential calculus are: one-step process, presence of analytical expression for $\Phi(\bar{u})$, differential properties of this function according to all variables, even if once, and usually the absence of limitations.

Examples of the use of differential calculus are given in § 4.5, 5.2, 5.3, 5.4, 6.1, 6.2, 6.3, 6.5, 7.1, 7.2, 7.3 and 7.4.

3.3. The Use of Calculus of Variations to Find Optimal Programs

Calculus of variations [19] considers functionals, by which are understood functions in which the role of independent variable is played by functions. Calculus of variations makes it possible to find these functions, maximizing or minimizing the functional (in our case — optimal programs).

Let

$$K = \int_{t_0}^{t_{кон}} f(t, u, \dot{u})\, dt, \tag{1}$$

where t — independent variable; $u(t)$ — function (control) which must be found from condition $\min K$ or $\max K$.

A more common statement is Meyer-Boltz [40], when

$$K = \int_{t_0}^{t_{кон}} f(t, u, \dot{u})\, dt + F[u(t_0);\ t_0;\ u(t_{кон});\ t_{кон})]dt.$$

We shall consider functional (1) later.

It is not difficult to see that calculus of variations is a generalization of differential calculus methods to the case of an infinite number of variables (function $u(t)$ can be considered as an infinite number of variables). If differential calculus makes it possible to determine the extremum of a function with a finite number of variables and to find the values of arguments (optimal parameters of control) with which these extremums are achieved, then calculus of variations makes it possible to calculate the functions (optimal function of control) with which the extremum achieves the functional in question.

Classical variational calculus considers only those problems without limitations on the function to be determined.

Problems of variational calculus can be classified as follows.

By the number of independent variables, problems are divided into problems with functions with one, two and many variables. Expression (1) is written for the case of a function with one variable. In the case of a function with two variables, the expression for K takes the following form:

$$K=\iint\limits_{\blacksquare} f\left[t_1, t_2; u(t_1, t_2); \frac{\partial u(t_1,t_2)}{\partial t_1}, \frac{\partial u(t_1,t_2)}{\partial t_2}\right] dt_1 dt_2. \qquad (2)$$

By the number of sought functions (controls), problems are divided into problems with one and many functions u.

By kind of functional, problems are divided into problems with a functional where the subintegral expression depends on t, u, $\dot{u}$, when the number of derivatives of u can vary ($\ddot{u}$, $\dddot{u}$, etc.) and problems with a degenerated functional, in which t, u or derivatives of u are absent. In the case of an incomplete functional, solution sometimes is greatly simplified.

By the kind of boundary conditions are distinguished problems with fixed end points, in which conditions $u(t_0)=A$, $u(t_{\text{кон}})=B$, must be met, problems with a sliding end point, in which end points lie on two fixed lines $u=\varphi(t)$ and $u=\psi(t)$, and problems with free end points.

Besides boundary conditions, subsidiary conditions can be superimposed or not. In particular, these can include subsidiary condition $l=\int_{t_0}^{t_{\text{кон}}} G(t; u, \dot{u})$ (so-called isoperimetric problem) or conditions of the $g_i(t, u)=0$, kind, i.e. sub-

sidiary functional connections.

Table 3.3.1 gives necessary conditions of existence of extremum of a functional for different variational problems.

Sufficient conditions for the existence of an extremum are complex, even for the simplest problems; therefore we shall not dwell on them.

Besides the method of solving variational problems described above, consisting of reducing a variational problem to analysis of a differential equation (or system), i.e. the indirect method, there are also direct methods of solving variational problems which in a number of cases are more effective than the indirect.

The essence of direct methods consists of:

— constructing a minimizing sequence of curves $u_1, u_2, \ldots, u_n,$, so that $\lim_{n\to\infty} K(u_n)=\mu;$

— proving that in this sequence there is limiting curve $u^{(0)}$;

— proving the validity of the limiting process

$$K(u^{(0)})=\lim_{n\to\infty} K(u_n). \tag{3}$$

There are different direct variational methods, differing from each other in methods of constructing minimizing sequences of functions.

The Ritz method consists of selecting a certain sequence of functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ in such a way that the functions themselves and their linear combinations $u_n = c_1\varphi_1 + c_2\varphi_2 + \ldots + c_n\varphi_n$ are permissible for the functional. The the problem is reduced to finding the minimum of a function with n variables $(c_1, c_2, \ldots, c_n)$, , which is much simpler than the problem of finding the minimum of the functional.

Thus, calculus of variations makes it possible to solve problems of mathematical programming if the model is described by analytical functions, continuous and differential, and if the problem does not contain limitations on controls and dependent variables (coordinates). An example of the use of calculus of variations is given in § 7.7.

TABLE 3.3.1

Kind of problems	Necessary conditions from which is found $u(t)$
$\int_{t_0}^{t_{\text{кон}}} f(t,\, u,\, \dot{u})\, dt;$ $u(t_0) = A;$ $u(t_{\text{кон}}) = B.$	Euler's equation $\frac{\partial f}{\partial u} - \frac{d}{dt}\left[\frac{\partial f}{\partial (du/dt)}\right] = 0$ a)
$\iint_{\Omega} f\left(t_1,\, t_2;\, u;\, \frac{\partial u}{\partial t_1},\, \frac{\partial u}{\partial t_2}\right) dt_1 dt_2$ At boundary Ω and $(t_1,\, t_2)$ take given values	Euler's equation $\frac{df}{\partial u_K} - \frac{\partial}{\partial t_1}\left\{\frac{\partial f}{\partial (\partial u/\partial t_1)} - \frac{\partial}{\partial t_2}\left[\frac{-\partial f}{\partial (\partial u/\partial t_2)}\right]\right\} = 0$ b)
$\int_{t_0}^{t_{\text{кон}}} f(t;\, u_1,\, u_2,\, \ldots,\, u_K;\, \dot{u}_1;\, \dot{u}_2,\, \ldots,\, \dot{u}_K)\, dt;$ $u_1(t_0) = A_1;\; \ldots;\; u_K(t_0) = A_K;$ $u_1(t_{\text{кон}}) = B_1;\; \ldots;\; u_K(t_{\text{кон}}) = B_K$	Euler's equation $\frac{\partial f}{\partial u_K} - \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial (du_K/dt)}\right] = 0$ with $k = 1,\, 2,\, \ldots,\, K$ a)
$\int_{t_0}^{t_{\text{кон}}} f(t;\, u,\, \dot{u},\, \ldots,\, u^{(n)})\, dt;$ $u(t_0) = A;\; \ldots;\; u^{n}(t_0) = A^{(n)};$ $u(t_{\text{кон}}) = B;\; \ldots;\; u^{n}(t_{\text{кон}}) = B^{(n)}$	Euler-Poisson equation $\frac{\partial f}{\partial u} - \frac{d}{dt}\left\{\frac{\partial f}{\partial (du/dt)} + \ldots + (-1)^n \frac{d^n}{dt^n} \times\right.$ $\left.\times\left[\frac{\partial f}{\partial (d^n u/dt^n)}\right]\right\} = 0$ a)
$\int f(t,\, u,\, \dot{u})\, dt;\; u(t_0) = A;\; u(t_{\text{кон}}) = B;$ $\int_{t_0}^{t_{\text{кон}}} G(t,\, u,\, \dot{u})\, dt = l.$ Isoperimetric problem	$\frac{\partial f}{\partial u} - \frac{d}{dt}\left[\frac{\partial f}{\partial (du/dt)}\right] + \lambda\left\{\frac{\partial G}{\partial u} - \frac{d}{dt}\left[\frac{\partial G}{\partial (du/dt)}\right]\right\} = 0$ c)
$\int_{t_0}^{t_{\text{кон}}} f(t;\, u_1,\, u_2,\, \ldots,\, u_K;\, \dot{u}_1,\, \dot{u}_2,\, \ldots,\, \dot{u}_K)\, dt;$ $u_1(t_0) = A_1;\; \ldots;\; u_K(t_0) = A_K;$ $u_1(t_{\text{кон}}) = B_1;\; \ldots;\; u_K(t_{\text{кон}}) = B_K;$ $g_1(t;\, u_1,\, u_2,\, \ldots,\, u_K) = 0;$ $g_L(t;\, u_1,\, u_2,\, \ldots,\, u_K) = 0.$ **Lagrange problem**	With $K = 2$ and $L = 1$ $\begin{cases} \frac{\partial f}{\partial u_1} + \lambda \frac{\partial g}{\partial u_1} - \frac{d}{dt}\left[\frac{\partial f}{\partial (du_1/dt)}\right] = 0; \\ \frac{\partial f}{\partial u_2} + \lambda \frac{\partial g}{\partial u_2} - \frac{d}{dt}\left[\frac{\partial f}{\partial (du_2/dt)}\right] = 0. \end{cases}$ a)

Continuation of TABLE 3.3.1.

Kind of problems	Necessary conditions from which is found $u(t)$
$\int_{t_0}^{t_{\text{кон}}} f(t, u, \dot{u})\, dt$ End points of extremes lie in curves $u = \varphi(t);$ $u = \psi(t).$ Problem with sliding end points	$\frac{df}{du} - \frac{d}{dt}\left[\frac{\partial f}{\partial (du/dt)}\right] = 0;$ $f + \frac{\partial f}{\partial (du/dt)}\left(\frac{d\varphi}{dt} - \frac{d\psi}{dt}\right)\Big\|_{t=t_{\text{кон}}} = 0;$ $f + \frac{\partial f}{\partial (du/dt)}\left(\frac{d\varphi}{dt} - \frac{d\psi}{dt}\right)\Big\|_{t=t_0} = 0.$ Last two equations — equations of transversality

Note: a) Constants of integration are outside boundary conditions.
b) Constants of integration are determined from the condition that $u(t_1, t_2)$ take given values at boundary of area в.
c) Constants of integration and λ are found, based on boundary conditions and condition

$$\int_{t_0}^{t_{\text{кон}}} G(t, u, \dot{u})\, dt = l.$$

3.4. The Maximization Principle of L. S. Pongryagin. Discrete Maximization Principle

A further development and generalization of problems of variational calculus is the maximization principle of L. S. Pontryagin [41]. This principle makes it possible to solve problems with limitations on controls and with any connections between variables (including non-holonomic, i.e. including derivatives). This principle can be formulated as follows.

Let us have to find control

$$U(t) \in U(t_{\text{кон}}). \quad (1)$$

where $\overline{U}(t)$ — vector-function of control, characterized by components $u_k(t)$ and converting phase point from initial position $\overline{X}_0$ to final $X_{\text{кон}}$ (here $\overline{X}$ — vector of the position of the system, characterized by phase coordinates x_i) so that functional

$$K=\int_{t_0}^{t_{\text{кон}}} f_0\,[\overline{X}(t);\ \overline{U}(t)]\,dt \tag{2}$$

takes a minimal value;

$\overline{U}(t)_{\text{доп}}$ — area of permissible values of vector function of the control.

In this case there is a system of ordinary differential equations, describing the connections between phase coordinates and controls,

$$\dot{x}_i=g_i(\overline{X};\overline{U}),\ \text{where}\ \ i=1,2,\ldots,I. \tag{3}$$

If the right-hand sides of the equations depend explicitly on an independent variable (time), then we can introduce auxiliary phase coordinate $x_{I+1}=t$, and then $\overline{X}$ is converted to $(I+1)$-dimensional vector.

If t_0 and $t_{\text{кон}}$ are not fixed, this is a problem with unfixed time, which differs from the general statement only by lack of limitations on phase coordinates $(\overline{X}\in X_{\text{доп}})$.

Although theoretically the question of the possibility of calculating these limitations is also resolved, practical solution of problems in this case is difficult.

Use of the maximization principle requires formulating an auxiliary function of the following kind (type of Hamilton function):

$$H(\overline{\Psi},\overline{U},\overline{X})=\sum_{i=1}^{I}\psi_i g_i(\overline{X},\overline{U})+\psi_0 f_0(\overline{X},\overline{U}), \tag{4}$$

where $\overline{\Psi}=(\psi_0,\psi_1,\ldots,\psi_I)$ — auxiliary (associated) $(I+1)$-dimensional vector;

$$\dot{x}_0=f_0(\overline{X},\overline{U}). \tag{5}$$

Having taken partial derivatives of function H according to ψ_i and x_i, we obtain the following system of equations:

$$\dot{x}_i = \frac{\partial H}{\partial \psi_i}, \text{ where } i = 0, 1, 2, \ldots, l; \tag{6}$$

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i}, \quad i = 0, 1, 2, \ldots, l; \tag{7}$$

then

$$\dot{\psi}_i = -\sum_{i=0}^{l} \frac{\partial g_i}{\partial x_i} \psi_i, \quad i = 0, 1, 2, \ldots, l \tag{8}$$

According to the principle of maximization, a necessary condition for optimality of control $\overline{U}(t)$ and trajectory $\overline{X}(t)$ (in the sense of ensuring a minimum of functional K) will be the existence of the non-zero vector-function $\overline{\Psi}(t)$, corresponding vector function $\overline{U}(t)$ and $\overline{X}(t)$, at which:

a) for any t, belonging to segment $[t_0, t_{кон}]$, function $H[\overline{\Psi}(t), \overline{X}(t), \overline{U}(t)]$ of variable $\overline{U}$ from the area of permissible controls reaches at fixed point $\overline{U} = \overline{U}^*(t)$ maximum $M[\overline{\Psi}(t), \overline{X}(t)]$;

b) at end point $t = t_{кон}$ are fulfilled ratios

$$\psi_0(t_{кон}) = 0; \quad M[\overline{\Psi}(t_{кон}), \overline{X}(t_{кон})] = 0. \tag{9}$$

Calculations are conducted in approximately this sequence.

First function H of the Hamilton function type is formulated. Then, having set insufficient initial conditions at one end of the trajectory, the system of equations (5) and (8) is solved jointly, at each step selecting control $\overline{U}$ from condition $\max H$. The final values obtained are compared with the given values and the purpose of further calculations is to find corrections for initial approximate conditions, minimizing deviations between calculated final values and given values.

We have been describing the numerical method of solution. If a conjugate system must be integrated, then we obtain the solution in analytical form.

With certain modifications, we can also use this principle for solving discrete problems; in these cases it is called the discrete maximization principle [49].

Examples of the use of the maximization principle are given in § 4.4 and 6.5.

3.5. Methods Based on Sufficient Conditions
Methods of V. F. Krotov

In solving a series of variational problems taking into account limitations on controls and phase coordinates, the optimal control is not determined by using the principle of maximization. Thus, if controls enter a function of the Hamilton type linearly, condition

$$\max H = Au + B \tag{1}$$

with

$$u_{min} \leq u \leq u_{max}$$

leads to ratios

$$u = \begin{cases} u_{max} & \text{at } A > 0; \\ u_{min} & \text{at } A < 0. \end{cases} \tag{2}$$

In this case, when $A = 0$, The Hamiltonian does not depend on the control and the principle of maximization does not solve problems about optimal control.

For solving problems of this class, called degenerate variational problems, as well as for solving problems with impulse controls and sliding regimes, it is possible to use sufficient conditions of the absolute minimum, developed by V. F. Krotov [31].

The problem is formulated as follows. Find

$$\min K = F[X(t_0), X(t_{\text{кон}})] + \int_{t_0}^{t_{\text{кон}}} f_0[t, X(t), U(t)]\, dt \tag{3}$$

with

$$x = g(t, X, U), \tag{4}$$

where

$$X(t) \in X_{\text{доп}};$$
$$U(t) \in U_{\text{доп}}[t, X(t)];$$

$X_{\text{доп}}, U_{\text{доп}}$ — given sets.

The method consists of selecting sequences $\{X^s(t),\ U^s(t)\}$, in which is fulfilled the ratio

$$\lim K[\bar{X}^s, \bar{U}^s] = \inf L, \tag{5}$$

where inf (inferium) — exact lower boundary; s — number of approximation;

$$L = \Phi[\bar{X}(t_0), \bar{X}(t_{\text{кон}})] - \int_{t_0}^{t_{\text{кон}}} R[t, \bar{X}(t), \bar{U}(t)]\, dt; \tag{6}$$

$$R(t, \bar{X}, \bar{U}) = \sum_{i=1}^{n} \frac{\partial\varphi(t, \bar{X})}{\partial x_i} g_i(t, \bar{X}, \bar{U}) - f_0(t, \bar{X}, \bar{U}) + \frac{\partial\varphi}{\partial t}; \tag{7}$$

$$\Phi(\xi, \eta) = F(\xi, \eta) + \varphi(t, \eta) - \varphi(t_0, \xi), \tag{8}$$

here $\varphi(t, \bar{X})$ — continuous differential function, selected so as to fulfill condition (5).

The problem amounts to determination of function φ and minimization of the sequence from which must be fulfilled conditions

$$\Phi[\bar{X}^s(t_0), \bar{X}^s(t_{\text{кон}})] \to \inf F(\xi, \eta); \tag{9}$$

$$\xi \in \bar{X}_{\text{доп}}(t_0);\ \eta \in \bar{X}_{\text{доп}}(t_{\text{кон}});$$

$$R[t, \bar{X}^s(t), \bar{U}^s(t)] \to \sup R(t, \bar{X}, \bar{U}); \tag{10}$$

$$\bar{X} \in \bar{X}_{\text{доп}};\ \bar{U} \in \bar{U}_{\text{доп}}.$$

In solving ordinary problems, the role of function φ is played by potential V of a Bellman equation and generalized impulses Ψ of the maximization principle.

3.6. Dynamic Programming

A general statement of the problem of dynamic programming [15] is the following.

From the set of permissible controls $\bar{U}_{\text{доп}}$ find the control $\bar{U}$, which will convert a physical system from initial state $\bar{X}_0 \in \bar{X}_{0\,\text{доп}}$ to final $\bar{X}_{\text{кон}}$ $\bar{X}_{\text{кон доп}}$ so that a certain criterion $K(U)$ is maximized, i.e.

$$K^* = \max_{\bar{U}} \{K(\bar{U})\}.$$

Here the system must be found in the permissible area, i.e. $\bar{X} \in \bar{X}_{\text{доп}}$.

Thus, in problems of dynamic programming we consider:

— controllable system, i.e. a system which in time can change its state; there is the possibility of controlling this process;

— criterion K, which numerically expresses our interest in achieving particular results;

— initial state of system $\overline{X}_0$, limited by certain conditions $X_{0\,доп}$;

— final state of system $X_{кон}$, also limited by certain conditions $X_{кон\,доп}$;

— control, which can also have a number of limitations $U \in U_{доп}$.

Selecting a control means selecting a certain trajectory of a point in phase space, a definite law of movement. Fig. 3.6.1 illustrates a case of two phase coordinates.

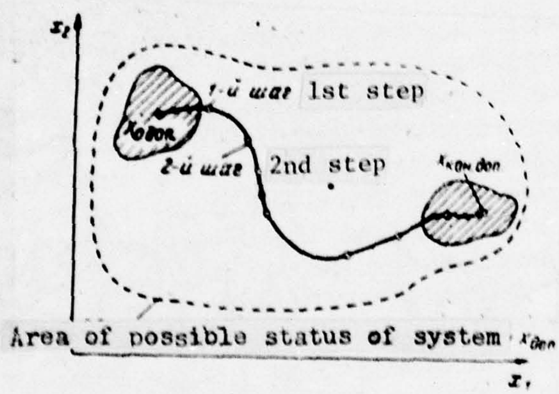

Figure 3.6.1. Phase space.

In terms of phase space, the problem of dynamic programming is formulated as follows.

Find the (optimal) control $\overline{U}$ under the effect of which point $\overline{X}$ of phase space is shifted from the initial area $\overline{X}_0$ to final area $X_{кон}$ without going beyond permissible area $X_{доп}$ so that criterion K is maximized.

The statement of a problem of dynamic programming does not differ essentially from the general statement of a mathematical programming problem.

For dynamic programming, which is a numerical method, the following example is typical: the process of transfer is divided into a number of sequential steps (conversion to discrete problem) and each one, beginning with the last, is subject to sequential optimization. A conditional optimal control is found at each step (with all possible assumptions about results of previous step), and then, when the processes reaches the initial state $\overline{X}_0$, the entire sequence of steps is iterated, but now from the set of conditional optimal controls, one is selected.

Thus, single solution of a complex problem is replaced by multiple solution of a simple problem.

The principle of optimality is used here. Optimal strategy has the following property: whatever the initial state and accepted initial solution, all remaining solutions in succeeding steps must comprise optimal strategy in relation to the state resulting from the first solution.

Requirements for the structure of processes analyzed by the dynamic programming method:

— a small number of phase coordinates;

— controllable Markov process, i.e. previous history has no value in determining future actions;

— the criterion has the property of additiveness, i.e. its value is determined by summation of partial values reached at individual steps (in a number of cases, a nonadditive criterion is reduced to additive form artificially, for example, by logarithmation of a derivative).

Typical of dynamic programming are problems in which steps are set by their own essence. However, in many cases, these steps are introduced formally and the distance between them selected as a compromise between requirements for accuracy and simplicity of solution.

An example of using the method is given in § 5.3.

3.7. Problems of Linear, Nonlinear and Integral Programming and Methods of Solving Them

A general statement of problems of linear programming [55] is the following. Find a set of variables $u_1, u_2, \ldots, u_K$, maximizing (minimizing) the function

$$K = \sum_{k=1}^{K} u_k C_k \tag{1}$$

with limitations

$$\sum_{k=1}^{K} a_{ik} u_k < b_i \text{ for } i = 1, 2, \ldots, m; \tag{2}$$

$$u_k > 0 \text{ for } k = 1, 2, \ldots, K. \tag{3}$$

Many methods have been developed for solving linear programming problems; they can be divided into two groups:

— finite, guaranteeing finding an accurate solution after a finite number of steps;

— interative, guaranteeing only an approximate solution of the problem.

The first can be divided into three groups:

— sequential improvement of the plan (simplex method);

— sequential reduction of discrepancies;

— combination.

Difficulties of solving problems are determined primarily by the value of derivative mK. At present, a BESM-6 electronic computer can be used to solve 250 x 250 problems in about one hour.

One direction in solving large size problems is block programming. It analyzes the possibility of solving large size problems by solving a number of partial problems with a small number of variables and limitations.

Frequently, the demand of integrality is made of values u_k, which is especially important if u_k is small (0, 1, 2), as with large u_k, rounding off does not cause significant errors.

Solution algorithms have been developed to solve integral programming problems, in particular the Gomory method. Solution of these problems is much more complicated than problems of ordinary linear programming; the increased expenditures of machine time can range from 10-15% to several times.

Gesides the Gomory method, problems of integral programming can be solved by the additive method of Balas.

In this case, if instead of condition (1) there is condition

$$K = \Phi(u_1, u_2, \ldots, u_K). \tag{4}$$

where requirements of linearity are not imposed on Φ , we get a problem of nonlinear programming. No single methods of solving such a problem have yet been developed. However, it can be solved in the following cases (there are algorithms for solving it on computers).

1. When function Φ is twice differentiated. Then all stationary points are found within the permissible area, they are verified as candidates for maximum, and the values of the target function at these points are compared with maximums achieved at the boundaries of permissible areas. The method of solving such a problem is extremely laborious.

2. When function Φ is concave and limitations are also concave functions (there are methods of solving problems, but they are very complicated).

3. If Φ is a polynomial of the second power in relation to u_k, and limitations are linear, this is called a quadratic programming problem. A machine method has been developed for solving such a problem.

3.8. Regular Methods of Finding the Extremum

Numerical methods of finding the extremum of a function with one or many variables $\Phi(\bar{U})$ [48], in which definite procedures are used (series of actions), are called regular in distinction from random search which will be discussed below. Limitations on the area of change of the argument are mandatory in this case.

Distinguished are passive search in which the results of preceeding steps are not taken into account (calculations of $\Phi(U)$), and sequential search, where these results are considered.

Regular search methods have been well developed for finding the extremum of a function with one variable, if this function is unimodal, i.e. in the interval of permissible change of values of a variable from 0 to $u_{\text{доп}}$ it has one extremum.

The criterion of effectiveness of search is a function of the interval of indeterminacy from the number of steps n. By the interval of indeterminacy is understood difference L_n of values of the variable between which finding the extremum is guaranteed.

The optimal procedure of passive search with an even number of steps (calculations of Φ) consists of the following. Values of $\Phi(u)$ are calculated at the following points:

$$\frac{u_k}{u_{\text{доп}}}=\frac{(1+\varepsilon)\left[\frac{k+1}{2}\right]}{\frac{n}{2}+1}-\left(\left[\frac{k+1}{2}\right]-\left[\frac{k}{2}\right]\right)\varepsilon, \tag{1}$$

where ε — the least distance between $\Phi(u)$ at which a difference can be found between $\Phi(u)$ and $\Phi(u+\varepsilon)$; [ ] — indicates the largest whole number, less than that enclosed in brackets or equal to it.

Taken as extreme is the greatest (least) value of $\Phi(u)$ and its corresponding u. The interval of indeterminacy in this case is equal to

$$L_n=u_{\text{доп}}\frac{1+\varepsilon}{\frac{n}{2}+1}, \tag{2}$$

where n — number of steps.

Let us note that we also use passive search in those cases when $\Phi(u)$ does not have the property of unimodality.

In this case, interval $u_{\text{доп}}$ is divided into different segments, $\Phi(u_k)$ is calculated with

$$u_k=\frac{k}{n-2} \tag{3}$$

and the greatest (least) value of $\Phi(u)$ and its corresponding u are selected.

Then the interval of indeterminacy is

$$L_n = u_{доп} \frac{1+\varepsilon}{\frac{n}{2}-1}, \tag{4}$$

i.e. slightly higher than in the preceeding case.

The optimal procedure of sequential search for a maximum for unimodal functions consists of calculating two values $\Phi(u_{11})$ and $\Phi(u_{12})$, where u_{11} and u_{12} are selected according to certain rules, which we shall discuss below, based on interval $i_1 = u_{доп}$. Depending on the obtained values of $\Phi(u_{11})$ and $\Phi(u_{12})$, a new interval is selected according to rules:

if $\Phi(u_{11}) > \Phi(u_{12})$, then $i_2 = \left[0, \left(u_{12} + \frac{\varepsilon}{2}\right)\right]$;

if $\Phi(u_{11}) < \Phi(u_{12})$, then $i_2 = \left[\left(u_{11} - \frac{\varepsilon}{2}\right), u_{доп}\right]$;

if $\Phi(u_{11}) = \Phi(u_{12})$, then $i_2 = \left[\left(u_{11} - \frac{\varepsilon}{2}\right), \left(u_{21} + \frac{\varepsilon}{2}\right)\right]$.

An analogous procedure is repeated in the next j-step, based on the new interval, etc.

Depending on the order of selection of u_{j1} and u_{j2}, we distinguish the dichotomy, Fibonacci, "golden section," and finally the search by discrete methods.

In the dichotomy method (half division)

$$u_{j_1} = \frac{i_j}{2} - \frac{\varepsilon}{2};$$

$$u_{j_2} = \frac{i_j}{2} + \frac{\varepsilon}{2} \tag{5}$$

and the interval of indeterminacy is

$$L_n = \left[2^{-\frac{n}{2}} + \left(1 - 2^{-\frac{n}{2}}\right)\varepsilon\right] u_{доп}. \tag{6}$$

In the Fibonacci method, the distance from the end of the interval is determined from ratio

$$l_{n-k}=F_{k+1}l_n-F_{k-1}\varepsilon, \tag{7}$$

where

$$l_n=\frac{u_{\text{доп}}}{F_n}+\frac{F_{n-2}}{F_n}\varepsilon; \tag{8}$$

F — Fibonacci number,determined by the following ratios:

$$F_0=F_1=1;$$

$$F_k=F_{k-1}+F_{k-2},\ k=2,3,\ldots$$

Let us note that the use of these formulae requires knowledge of the number of steps n which will be made.

In the "golden section" method, distances from the end of the interval at each step are calculated from the ratio

$$\frac{l_{k-1}}{l_{k+1}}=\tau^2,$$

where

$$\tau=\frac{1+\sqrt{5}}{2}=1{,}618033989,\ldots$$

In this case

$$L_n=\frac{u_{\text{доп}}}{\tau^{n-1}}. \tag{9}$$

There are also other methods of search: the method of sequential division of a segment (division of a segment into 3-4 parts with subsequent calculation of values of the function at these points; finding the segment in which the extremum is located; repetition of the procedure in this segment, etc.), the method of quadratic interpolation (replacement of the studied function in the segment in the extremum area by a quadratic parabola).

In the first case the interval of indeterminacy is

$$L_n=2u_{\text{доп}}\exp\left(-\frac{n-2}{2{,}2}\right). \tag{10}$$

Finally, sometimes u can take discrete values. In this case, discrete points must be distributed in a continuous interval so that the values of argument, determined by the Fibonacci method, coincide with these discrete points.

In conclusion, we present Table 3.8.1., in which the effectiveness of these methods is compared, i.e. ratio of the initial interval to the interval of indeterminacy (with $\varepsilon = 0$).

TABLE 3.8.1

No. of calculated points	Methods						
	Passive search		Dichotomy	Fibonacci	"Golden section"	Sequential division of segment	Search by discretes
	arbitrary	unimodal					
5	1,5	3,5	4	8	6,85	1,9	12
10	4	6	32	89	76,0	16	143
15	6,5	8,5	128	987	843	186	1596
20	9	11	1024	10946	9349	1750	17710
24	11	13	4096	75025	64078	17300	121392

The table shows the high effectiveness of the Fibonacci method and close behind, the "golden section" method. In addition, it can help decide how important it is to have information on the function in question.

If there is no basis for considering the function unimodal, there remains one method — passive sampling, whose effectiveness is very low. The possibility of being limited to analyzing discrete values of u markedly increases the effectiveness of the search.

The problem is very severely complicated in those cases when an extremum of functions with many variables must be found. No universal effective methods, guaranteeing finding a global maximum (minimum), exists in these cases.

Then the following methods are used.

S c a n n i n g m e t h o d (blind search) consists of sequential sorting of all possible values of controls and recalling the highest value of the optimized criterion. This method is so laborious that it can be used only with a small number of variables, or in a case when variables can accept only discrete values, or with the use of analog computers.

An advantage of this method is the possibility of finding a global extremum regardless of the kind of function.

Coordinate ascent (descent) method, also called the Gauss-Seidel method, consists of sequential, step-by-step optimization for each variable. Its advantage is simplicity of algorithm.

The gradient method consists of calculating partial derivatives of the criterion by all variables $\partial K/\partial u_k$, calculation of the direction of the gradient and taking a step in this direction (if a maximum is sought). Then this procedure is repeated.

Practically, at n-point are calculated $\Delta K_{k,n}$ — the increment of the criterion with Δu increase of all controls. Coordinates of the next point are calculated by equations

$$U_{k,n+1} = U_{k,n} + \lambda \Delta K_{k,n}, \tag{11}$$

where λ — parameter characterizing the value of the step.

Method of fastest rise (descent) is a development of the gradient method and differs from it in that tradient $\Delta K_{k,n}$ is not calculated at each point. After calculation of the gradient, steps are continued in this direction as long as K increases (decreases). When increase (decrease) of K stops, the gradient is calculated at the corresponding point and the procedure repeated.

Method of exclusion with tangents to the level line with regard to the function of two variables consists of the following. The level line is calculated, i.e. values of the criterion with a fixed variable. Then a tangent to this line is drawn and the area from the tangent up to the other side of the level line is excluded from consideration. Sequentially repeating this procedure, the area in which the maximums can be found is reduced.

There are many other methods; however none, with the exception of scanning, guarantee finding a global extremum.

Examples of the use of regular methods of search are given in § 6.2-6.4, 7.1, 7.4 and 7.7.

3.9. The Use of the Random Search Method

The random search method [42] can be used to find the extremum of a function (functional) with many variables with any limitations. The use of special algorithms gives the random search method a global character. In this case, the method makes it possible to find an absolute (nonlocal) extremum with a certain probability.

The method does not depend on the kind of optimized function and can be used even when this function cannot be described. Such situations arise, for example, in the case of describing a process with a statistical model, determining the value of a function by experiments or in problems of the "black box" type.

The advantages of the random search method over regular methods of search grow with increased complexity (dimension) of the functional (process) and as the required accuracy of finding the extremum is reduced. The latter is of special interest in using the random search method in operations research, where complex optimization problems of an extremely approximate nature are often solved.

The essence of the method consists of the following.

We have to find the extremum of a certain function $\Phi(\bar{U})$, where $\bar{U}(u_1, u_2, \ldots, u_K)$ certain K-dimensional vecture, unequivocally determining the numerical value of function $\Phi(\bar{U})$. Let the minimum of this function be sought for determinacy.

We introduce the concept of a single K-dimensional random vector $\bar{\xi}$, individual realizations of which ξ_i are equally probably distributed in all K-dimensional space:

$$\bar{\xi} = (\xi_1, \xi_2, \ldots, \xi_K), \tag{1}$$

where $-1 < \xi_k < 1$ and normalized in a quadratic sense to one:

$$\sum_{k=1}^{K} \xi_k^2 = 1. \tag{2}$$

Two basic varieties of the random search method are distinguished: blind and sequential.

B l i n d s e a r c h is carried out very simply. With each ste p of the search, as the vector of variables is taken random vector

$$\bar{U}_n' = a\bar{\xi}_n \tag{3}$$

where a — vector of the scale determined by the possible limits of change in the vector of variables.

Here two cases are possible. If we do not know the minimum of function $\Phi_{min}(\bar{U})$, then after a rather large number of steps n, from all sets of numerical values of the function is selected the least, which is taken as the sought minimum. If the minimal value of the function is known, then the search is stopped when the following condition is fulfilled

$$|\Phi_{min}(\bar{U}) - \Phi(\bar{U}_n)| < \varepsilon, \tag{4}$$

where ε — some present value, determining the accuracy of the solution.

In the latter case, vector $\bar{U}_n$ is taken as the sought vector of variables, determining with a given accuracy the minimum of function Φ. .

It must be noted that blind search is of limited use because of significant losses of time. One of the conditions of using this kind of search is the possibility of variables taking only values of 0 or 1.

S e q u e n t i a l s e a r c h uses the idea of sequential improvement of the solution. This kind of search has a higher rate of convergence because its success is determined at each random step. A successful step is one where the numerical value of the function is less than the value at the preceeding successful step, otherwise the step is called unsuccessful. The determining factor is that all random steps are made from the last successful condition.

Simple and adaptive sequential methods of search are distinguished. In simple search, a random step is formed, taking into account information on the behavior of the minimized function, obtained only at the current step. All information in this case is exhausted only by establishing the success or non-success of the step.

From a large number of algorithms of simple random search we present only one, which is called "step-by-step search with recomputation."

The formula of this algorithm has the form:

$$\Phi_{n+1}=\Phi(\bar{U}_n+\Delta\bar{U}_{n+1}), \qquad (5)$$

$$\Delta\bar{U}_{n+1}=\begin{cases} a\bar{\xi}, \text{ if } \Phi_n<\tilde{\Phi}_{n-1} \text{ (successful step)}; \\ -\Delta\bar{U}_n+a\bar{\xi}, \text{ if } \Phi_n>\tilde{\Phi}_{n-1} \text{ (unsuccessful step)}; \end{cases}$$

where

$$\tilde{\Phi}_n=\min\Phi_j,\ j=1,2,\ldots,n.$$

This algorithm is called a step algorithm, because vector $\bar{U}$ changes with a certain random finite step $\Delta\bar{U}$. Step algorithms are most often used for function Φ, set in discrete form. Recalculation here is that in the case of an unsuccessful step, shift of $\Delta\bar{U}$ vector is represented formally in the form of the sum of a random "correction" $a\bar{\xi}$ and shift of the vector, obtained in the preceeding step with the opposite sign, i.e. actually the step is conducted from the last successful condition.

Continuous search algorithms are most often used for continuous functions. A characteristic feature is evaluation of the success of the step by the sign of derivatives of function Φ, in connection with the fact that random shifts are set not only by vector $\bar{U}$, but also by its derivative. Continuous algorithms are realized in analog computers (electronic models).

A d a p t i v e (learning, accomodative) search is distinguished by formation of a random step, taking into account information about the behavior of function Φ throughout all previous steps. This information is used in two directions: in reorganization of probability characteristics, determining the direction of a random step in space of variables, and in changing the scale of the step.

Introduced in adaptive algorithms is a certain function $\bar{W}$, representing the so-called "vector of memory." This vector is calculated at each step of the search according to various recurrent dependences. Changes in the scale of the step can conveniently be connected functionally with a certain characteristic x, reflecting the degree of approximation of function Φ to the sought extremum.

In general form, the expression for shift $\Delta\bar{U}$ can be represented as

$$\Delta\bar{U}_n=a\bar{\eta}_n, \qquad (6)$$

where

$$\bar{\eta}_n=\varphi(\bar{\xi},\bar{W}_n). \qquad (7)$$

For practical calculations it is convenient for α and x to take the following dependences:

$$\alpha = \frac{\alpha_0}{x^\gamma}. \tag{8}$$

The index of power γ can vary from 1 to 3, depending on the kind of function Φ.

$$x = \frac{n}{r},$$

where n — current number of steps; r — number of successful steps.

In practical use of algorithms of sequential random search, it is necessary to introduce a certain condition of termination of the process (calculation). Preset numerical value x can be used as this condition.

The experience of using random nonadaptive search to find extrems of a number of functionals shows that an average accuracy on the order of 1% can be achieved with x = 5-10.

In conclusion, we must note that random search is one of the newest methods and, therefore, many unsolved questions can arise in using it. However, the simplicity of the idea, its universality and ease of realization in machine algorithms, indicate that it is completely acceptable for solving a wide range of problems of mathematical programming.

Examples of the use of random search are given in § 4.3, 5.4 and 5.5.

3.10. Heuristic Programming

By heuristic programming is understood a wide range of problems connected with the use and modeling of processes of human thought, primarily the most delicate nuances of human behavior, arising in controlling increasingly complicated processes and objects.

Heuristic programming, from the point of view of operations research, is of interest in two directions:

— the use of results of heuristic programming to describe the actions of people in constructing operations research models;

— the use of heuristic programs for solving optimization problems arising in operations research.

In relation to problems of mathematical programming, heuristic programming can be represented in the following directions:

1. Replacement of one model with another. In this case, if the model cannot be studied by existing methods of mathematical programming, it can be replaced by another model, formulated on the basis of the experience of experts. This model will not strictly reflect the essence of the process in question, but it can be analyzed.

2. Compression of the area of study. In searching for an optimal probram based on a "strict" model, experts can impose additional limitations on the area of permissible solutions (besides those based on experience and intuitive reasoning) and thus significantly constrict this area, thereby facilitating the search for an optimal program. Limitations can be expressed both in the form of setting an aera of permissible solutions and in the form of introducing logical rules connecting the variables.

3. Setting up a "support plan" by experts, i.e. a program which from their point of view is optimal. Here the problem of study is facilitated, as in the case of a successful setting up of the program it is sufficient to verify change in the criterion of effectiveness only in its proximity.

4. Use of intuitive, nonstrict methods of finding optimal solutions, developed in the everyday practice of experts.

We shall illustrate the first direction by compiling one variant of a chess program.

If composition of the chess program is approached from the position of the classical solution of optimization problems, then we should consider all variants which could follow each possible move until the end of the game and select that move which would lead to the quickest win or most delay a loss. However, it turns out that in this case it would be necessary to consider 10^{120} variants, impossible on any computer.

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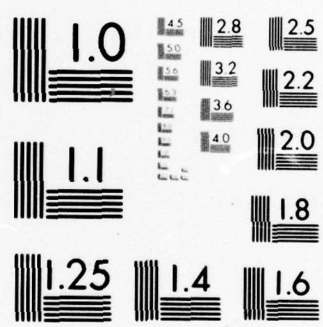

MICROCOPY RESOLUTION TEST CHART
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This means we must reject the path of strict solutions and create a program which would find a solution, even though not an optimal one in the strict sense of the word.

For this, we must first of all construct an estimator $f(P)$ of the chess position, using the experience of chess experts (i.e. nonstrict function), depending on the number of pieces the opponent has. The estimator can be essentially developed by means of more detailed calculation of pawn structure, the position of protective and attack pieces and the right of selection. But be that as it may be, it is obtained with the help of the estimates of experts and can be updated during the game.

Later, from a number of possible moves is selected one which should maximize change of the position estimator under the condition that the opponent is trying to minimize it. Existing machines can consider position only three moves ahead (and not to the end, i.e. not strictly).

This means that one more heuristic rule must be introduced, according to which individual variants must be considered in great depth. Conditions determining these variants(for example, attack on the opponent's pieces, primarily the king) are also set by the experts.

For further improvement of chess programs, one can introduce into the external memory of the machine an adequate number of openings, a number of standard positions, combinations and maneuvers, mastered with many practical games.

Thus,heuristic programming differs in a number of features. It is not strict and does not guarantee obtaining the best solution. Used in compiling a heuristic program is the experience of experts in a given area, formalized in the form of certain rules, empirical dependences and calculation schemes. Heuristic programming provides a method of finding a solution (not necessarily optimal) in those cases when classical methods of programming are weak.

The use of heuristic programming in certain problems of operations research can be extremely promising. These are primarily problems of large size, the solution of which takes a large number of experts (their experience will produce specific heuristic rules, general ideas for compiling heuristic programs). These processes are primarily those of designing technical devices.

Actually, in designing any technical device an enormous number of variants is possible, carried out according to various plans. The experienced designer, as a rule, immediately rejects the majority of them on the basis of his own experience.

Widely used in design are different kinds of heuristic rules and formulae, based on the experience of the preceeding design. Many examples of such rules and formulae could be presented. Suffice it to say that the experienced designer often makes his final evaluation of a design on the basis of the criterion "it checks out" (i.e. is a certain combination of ratios observed) or "it doesn't check out."

Explanation of these rules could greatly simplify technical device design units.

3.11. Stochastic Programming

We have been considering cases when all values and functions encountered in solving problems of mathematical programming were nonrandom (determined). However, in practice, in operations research problems we encounter random values (functions), i.e. we must solve problems of stochastic programming which are now in the first stage of development.

Usually in solving such problems, mathematical expectation of all values is considered which, ultimately, is not strict. In other cases, mathematical expectation of a criterion is calculated using probability theory methods, most often statistical tests, and this criterion is maximized (or minimized), varying with controls. Difficulties arising in calculations will be discussed below.

Some successes in solving stochastic problems of mathematical programming have been noted in the area of linear programming. In the usual case (3.7.1)-(3.7.3), C_k, a_{ik} and b_i are random values. The following indexes of quality can be considered:

a) mathematical expectation of the criterion;

b) mathematical expectation of the square of the criterion;

c) probability of the criterion exceeding a certain threshold or threshold value, which must exceed the criterion with a given probability;

d) mathematical expectation of the function of usefulness of the criterion.

The control (program, plan) can be considered as a combination of determined values or as a combination of random values. In the first case, a one-step plan can be determined in which conditions (3.7.2) and (3.7.3) are satisfied in all possible cases, or a two-step plan, consisting of a support plan and a correcting plan, determined depending on specific realizations of C_k, a_{ik} and b_i.

Methods of solving stochastic problems of convex nonlinear programming have now been developed in which is found the random vector of control

$$\bar{U}=\Phi(\bar{C},\bar{a},\bar{b}). \tag{1}$$

Sometimes the value of the criterion is determined with random errors. This can occur when the criterion is calculated with a statistical model or by means of direct experiment or approximation methods.

In this case, we must solve the problem under "interference conditions" or "with noise." The presence of errors makes the process of finding the extremum difficult.

Let us describe one of the procedures for solving such a problem using the regular search method for a case with one variable.

Coordinates of the center of the next (k+1)-pair of samples in this case are determined according to the formula

$$u_{k+1}=u_k+\frac{a_k\,[K(u_k+c_k)-K(u_k-c_k)]}{c_k}, \tag{2}$$

where a_k and c_k are functions of k and must be satisfied by the following conditions:

$$a_k\to 0 \text{ and } c_k\to 0 \quad \text{with } k\to\infty;$$

$$\sum_{k=1}^{\infty} a_k=\infty; \tag{3}$$

$$\sum_{k=1}^{\infty}\left(\frac{a_k}{c_k}\right)^2<\infty. \tag{4}$$

In particular, we can use dependences of the form

$$a_k=\frac{1}{k}; \tag{5}$$

$$c_k=\frac{1}{k^{0,4}}. \tag{6}$$

This series of actions can also be used in a multi-dimensional case.

3.12. Comparison of Different Methods of Mathematical Programming

The general principles of selecting a method of mathematical programming amount to the following. The use of analytical methods is always preferable to numerical, as in this case the effect of different factors on the optimum solution can always be studied, it can be used to solve related problems, and finally to obtain a sufficiently complete picture of the question being studied.

However, in solving practical problems of operations research, analytical solutions cannot always be obtained. Therefore, numerical methods must be used.

Information on the studied process and the possible character of the optimal solution must be carefully collected and utilized; this can significantly facilitate solution.

There is still no one mathematical method which is better than all the rest for solving all types of problems.

The most common methods, suitable for determining optimal controls (functions), are the principle of maximization and dynamic programming.

The maximization principle and dynamic programming can be derived mathematically from each other; however, they use qualitatively different ideology in solving problems.

In dynamic programming the study is conducted by searching the entire grid of I variables in one step, filling these variables in the corners of the grid and then selecting the optimal path between steps.

In using the maximization principle, first of all the optimal path is calculated, with subsequent improvement by satisfaction of boundary conditions.

From this it is clear that realization of possibilities of dynamic programming is connected with limitations of machine memory. This leads to a limited number of

nodal points and interpolation errors for intermediate values, impossible to evaluate.

The solution of problems using the principle of maximization is not connected with any limitations of machine memory. However, difficulties arise here, determined by the necessity of solving an extreme problem.

In some cases, when no solution can be found using dynamic programming, the use of the discrete maximization principle will solve the problem.

However, dynamic programming also has a number of important advantages:

1. Optimization of processes with limitations on variables of condition only simplifies solution of the problem by dynamic programming methods. At the same time, the principle of maximization still does not have available constructive methods of solving these problems.

2. Dynamic programming can be used for controlling processes whose conversion at each step cannot be described by equations and is, therefore, described by words. We apply the principle of maximization only to processes with well-determined conversion equations; conversion functions must be continuous and differential according to variables of condition.

3. Dynamic programming provides a global search, while the principle of maximization can, in some cases, give local optimums.

Sometimes dynamic programming is used first to find an area of a global extremum, and then the point of the extremum is more accurately defined with the aid of the principle of maximization.

One of the most successful directions in the development of dynamic programming is the use of properties of the processes being studied to reduce the volume of variants examined.

A connection has now been established between dynamic programming and the discrete principle of maximization, on one hand, and problems of linear, nonlinear and integral programming, on the other.

TABLE 3.12.1

Methods	Features of methods							
	Numerical (N) Analytical (A)	Requirements for function of criterion	Calculation of limitations according to coordinates X and controls U	Kind of control	Presence of connections between controls and coordinates	Necessary and sufficient conditions of extremum	Guarantee of finding global extremum	Feasibility of use
Differential calculus	A N	Analytical expression and differentiality	Impossible	System of values	Perhaps	Proven	—	
Calculus of variations	A N	Same	Same	Functions	Same	Sufficient conditions not always proven	—	
Principle of maximization	A N	"	Possible by U	"	"	Not proven	—	With large number of variables
Methods based on sufficient conditions	A N	"	Possible by U and X	"	"	Proven	—	With degenerated problems
Dynamic programming	N	No	Same	System of values	"	—	Yes	With small number of variables
Linear programming	N	Linear function of control	Possible by U	Same	No	—	Yes	Feasible in all cases when problem is reduced to linear programming

Continuation of TABLE 3.12.1

Methods	Features of methods							
	Numerical (N) Analytical (A)	Requirements for function of criterion	Calculation of limitations according to coordinates X and controls U	Kind of control	Presence of connections between controls and coordinates	Necessary and sufficient conditions of extremum	Guarantee of finding global extremum	Feasibility of use
Regular search	N	No	Possible by U and X	System of values	No	—	No	With small number of controls
Random search	N	No	Same	Same	No	—	No	With large number of controls
Heuistic programming	A N	No	"	Functions	Perhaps	—	"	In cases when other methods are ineffective

These problems are sufficiently clearly formulated in dynamic programming and the discrete principle of maximization as variants of transport problems.

If the problem is successfully reduced to linear programming, it can be considered solved, as there are quite a few effective algorithms for solving it.

If search methods must be used, preference is given to regular methods when the number of variables is small, and to random search when the number of variables is large

In conclusion we present Table 3.12.1, which contains some information on methods of mathematical programming.

4. GENERALIZED PROBLEM OF THE REPLACEMENT OF TECHNICAL DEVICES

4.1. Statement of the Problem of Selecting Optimal Times for the Development of New Types of Technical Devices

Thanks to technical progress, new, increasingly improved technical devices are being created which are much more effective (productive) than the previous ones, and they are making it possible to perform a given volume of work with fewer of them.

However, the development of new types of technical devices and putting them into production require large expenditures, the number of each type of device produced is reduced and, as a result, the cost of each device increases.

It is very difficult to answer correctly the question of whether or not it is feasible to produce new types of technical devices and when to produce them. It can be done only by analyzing summary expenditures for their development, production and exploitation in the case of different orders of development of new types.

We shall attempt to give a mathematical formulation of this problem.

As a result of prediction and calculation of the required number of technical devices, it is possible to obtain function $N(t_{p_i}; t)$ of the demand for these devices. This function, as a rule, is ascending from current time t and descending from the time of conclusion of development of the given type t_{p_i}.

Analysis of processes of storage or exploitation of technical devices establishes the so-called function of survival $v(t_{p_i}; t - t_n)$, where t_n — time of production of a given technical device. This function is a mathematical expectation of the number of devices in good repair at time t, relative to the number of devices produced by time t_n.

Finally, methods of predicting cost allow us to obtain the dependence of cost on the number of technical devices of each type produced, which has the following form:

$$C_{\text{п}}(t_{p_i}; N_{\text{п}_i}) = C_0(t_{p_i}) N_{\text{п}}(t_{p_i}; t)^{\mu}. \tag{1}$$

where $N_{\text{п}_i}$ – number of devices of i type produced; $C_0(t_{p_i})$ and μ – coefficients.

The first statement of the problem is the following. Determine the optimal replacement policy (times of development of new types of technical devices), minimizing summary expenditures for given period T with the condition of guaranteeing given demands at any moment.

The opposite statement is also possible: determine the optimal replacement policy, maximizing satisfaction of demands with fixed expenditures. According to § 1.5, both statements of the problem are equivalent; therefore, in the future, only the first will be considered.

Let us note certain properties of replacement policy. We shall assume that after development, the new type of technical devices is produced immediately, production of the old type of devices having stopped. Actually, if production of the new type of devices is not begun, it is better to conclude their development later, after having obtained a more improved technical device. However, the truth of this assertion cannot yet usually be proven mathematically.

Let us show in a very simple example that there are conditions when the presence of the first property of replacement policy can be proven.

Let:

$$N(t_{p_i}; t) = N_0 \exp(at - bt_{p_i}); \tag{2}$$

$$C(t_{p_i}) = C_0 \exp(ct_{p_i}); \tag{3}$$

$$v(t_{p_i}; t - t_{\text{п}}) = 1; \ \mu = 1; \ k_x(t_{p_i}) = 0. \tag{4}$$

At moment of time t_2, let development of the new type of technical device be concluded, but not put into production, and let production of the old type of devices continue during time Δt.

In this case, change in expenditures can be represented as the following function of Δt:

$$\Delta C = \left(C_1 \frac{\partial N_1}{\partial t} - C_2 \frac{\partial N_2}{\partial t}\right) \Delta t =$$
$$= C_0 N_0 a \Delta t \exp(at_2) [\exp(c - b) t_1 - \exp(c - b) t_2]. \tag{5}$$

Completion of development at t_2 is feasible only if expenditures connected with the start of production of the new type of devices at this time are less than expenditures for production of the old type of devices (developed at time t_1).

This condition can be written with assumptions as follows:

$$C_1 N(t_1;\ T) > C_2 [N(t_2;\ T) - N(t_2;\ t_2)] + C_1 N(t_1;\ t_2) \tag{6}$$

or

$$C_1 [N(t_1;\ T) - N(t_1;\ t_2)] > C_2 [N(t_2;\ T) - N(t_2;\ t_2)],$$

from which

$$\begin{gathered} C_0 N_0 \exp(ct_1) [\exp(aT - bt_1) - \exp(at_2 - bt_1)] > \\ > C_0 N_0 \exp(ct_2) [\exp(aT - bt_2) - \exp(at_2 - bt_2)]; \\ \exp(ct_1 - bt_1) [\exp(aT) - \exp(at_2)] > \exp(ct_2 - bt_2) \times \\ \times [\exp(aT) - \exp(at_2)] \end{gathered}$$

or

$$\exp(c - b) t_1 > \exp(c - b) t_2. \tag{7}$$

Comparison with equality (5) shows that with any positive Δt, we obtain

$$\Delta C > 0,$$

i.e. delay of production of the new type of devices after their development will increase expenditures.

Therefore, having completed development of the new type of technical devices, it is not feasible to continue production of the old type of devices, and for the case in question, the first property of replacement policy can be considered proven.

The second evident property of replacement policy is the following. Development of the new type should begin at a time so that it will be completed and production of the new type of devices be begun before the end of the period in question.

These properties of replacement policy will be considered in a later analysis.

We shall first describe conditions of fulfilling all tasks at each moment of time. Let us designate by $N_x(t_{p_i}; t)$ the number of devices of i-type (i.e. the type

whose development was completed at time t_{p_i}) in working order at time t. Let us introduce the concept of "index of effectiveness of the technical device" ($\Im_i^j$) , the value of which depends on the time of development and the time of use of the device.

General productivity of technical devices

$$\Pi^j = \sum_{i=1}^{k} \Im_i^j N_x(t_{p_i};\ t_j).$$

The required productivity of technical devices can be expressed by the required number of devices of a given kind:

$$\Pi^j = \Im_1^j N(t_{p_1};\ t_j) = \ldots = \Im_i^j N(t_{p_i};\ t_j).$$

from which

$$\Im_i^j N(t_{p_i};\ t_j) = \sum_{i=1}^{k} \Im_i^j N_x(t_{p_i};\ t_j).$$

Having expressed $\Im_i^j$ by $\Im_1^j$, after substitution and reduction, we obtain

$$1 = \sum_{i=1}^{k} \frac{N_x(t_{p_i};\ t_j)}{N(t_{p_i};\ t_j)}. \tag{8}$$

Let us consider the connection between the number of devices produced and the number of devices suitable for exploitation. According to determination of the function of survival, we can write equation

$$N_x(t_{p_i};\ t) = \int_{t_{p_i}}^{t} \frac{dN_n(t_{p_i};\ t_n)}{dt_n} \nu(t_{p_i};\ t - t_n)\, dt_n. \tag{9}$$

Taking into account equation (9), we write equation (8) as follows:

$$\sum_{i=1}^{k} \frac{1}{N(t_{p_i};\ t)} \int_{t_{p_i}}^{t} \frac{dN_n(t_{p_i};\ t_n)}{dt_n} \nu(t_{p_i};\ t - t_n)\, dt_n = 1. \tag{10}$$

This is the final expression of the condition of fulfilling all tasks.

Equation (10) makes it possible with the given replacement policy (i.e. set of t_{p_i}) to determine $\frac{dN_n(t_{p_i};\ t_n)}{dt_n}$ for any time, and, integrating this function, to determine the volume of production of devices of all types $N_n(t_{p_i};\ t_n)$ in the function of time.

Summary expenditures C_Σ are composed of expenditures for development of all types of devices, for their production and storage or exploitation. For devices of one type, these expenditures up to time T can be written in the following form:

$$C_{\Sigma_i}=C_p(t_{p_i})+C_o(t_{p_i})N_n(t_{p_i};\,T)^{\mu}+ +C_x(t_{p_i})\int_{t_{p_i}}^{T}N_x(t_{p_i};\,t)\,dt \tag{11}$$

or

$$C_{\Sigma_i}=C_o(t_{p_i})\left[k_p(t_{p_i})+N_n(t_{p_i};\,T)^{\mu}+ +k_x(t_{p_i})\int_{t_{p_i}}^{T}N_x(t_{p_i};\,t)\,dt\right], \tag{12}$$

where

$$k_p(t_{p_i})=\frac{C_p(t_{p_i})}{C_o(t_{p_i})};\quad k_x=\frac{C_x(t_{p_i})}{C_o(t_{p_i})}.$$

Taking into account equality (9), expression (12) assumes the form

$$C_{\Sigma_i}=C_o(t_{p_i})\left\{k_p(t_{p_i})+\left[\int_{t_{p_i}}^{T}\frac{dN_n(t_{p_i};\,t_n)}{dt_n}dt_n\right]^{\mu}+ +k_x(t_{p_i})\int_{t_{p_i}}^{T}\int_{t_{p_i}}^{t}\frac{dN_n(t_{p_i};\,t_n)}{dt_n}\nu(t_{p_i};\,t-t_n)\,dt_n\,dt\right\}. \tag{13}$$

Summary expenditures for all types of devices, taking into account reduction of expenditures to a single moment of time (§ 1.6), can be written in the form of equation

$$C_\Sigma=\sum_{i=1}^{k}C_o(t_{p_i})\left\{\frac{k_p(t_{p_i})}{a^{(t_{p_i}-0.5\tau_p)}}+\left[\int_{t_{p_i}}^{T}\frac{\frac{dN_n(t_{p_i};\,t_n)}{dt_n}}{a^{t_n}}dt_n\right]^{\mu}+ +k_x(t_{p_i})\int_{0}^{T}\frac{1}{a^{t}}\int_{t_{p_i}}^{t}\frac{dN_n(t_{p_i};\,t_n)}{dt_n}\nu(t_{p_i};\,t-t_n)\,dt_n\,dt\right\}, \tag{14}$$

where

$$a=1+\alpha; \tag{15}$$

ε — coefficient of effectiveness of investments; τ_{p_i} — time used in developing technical devices of i-type.

Thus, mathematical formulation of this problem is as follows: determine the set $t_{p_1}, t_{p_2}, \ldots, t_{p_k}$, including the number of k (number of developments), which, taking into account condition (10), will ensure a minimum C_{Σ}, determined by equation (14).

Let us note some features of this problem. First of all, this is not a Markov process, as expenditures in a given year depend on prehistory (time of production of models). Therefore, the method of dynamic programming cannot be used to solve it.

Functional C_{Σ} does not have the property of additiveness, which introduces additional difficulties in solving this problem.

In the future, we shall assume that at the initial moment of time there is a sufficient number of technical devices, produced at that time, so we need not consider "catching up" with demands in detail, which represents a separate problem.

4.2. Solution of the Problem by the Complete Sorting Method

In the majority of cases encountered, the problem of selecting optimal times of development is by its nature discrete. As planning is conducted once a year, the natural discrete space is one year. Let us consider a discrete problem.

Every year two cases are possible: at the start of the year development of a new model is completed (we shall arbitrarily call this case 1) or not completed (we shall arbitrarily call it 0). Any possible policy can be described in the form of a binary number in which the number of places is equal to the number of years in the period under consideration. Graphically all possible replacement policies can be illustrated by a binary graph (Fig. 4.2.1).

The total number of possible replacement policies M, as it is not hard to see, is equal (taking into account properties of replacement policies) to

$$M = 2^{T-\tau_p}. \quad (1)$$

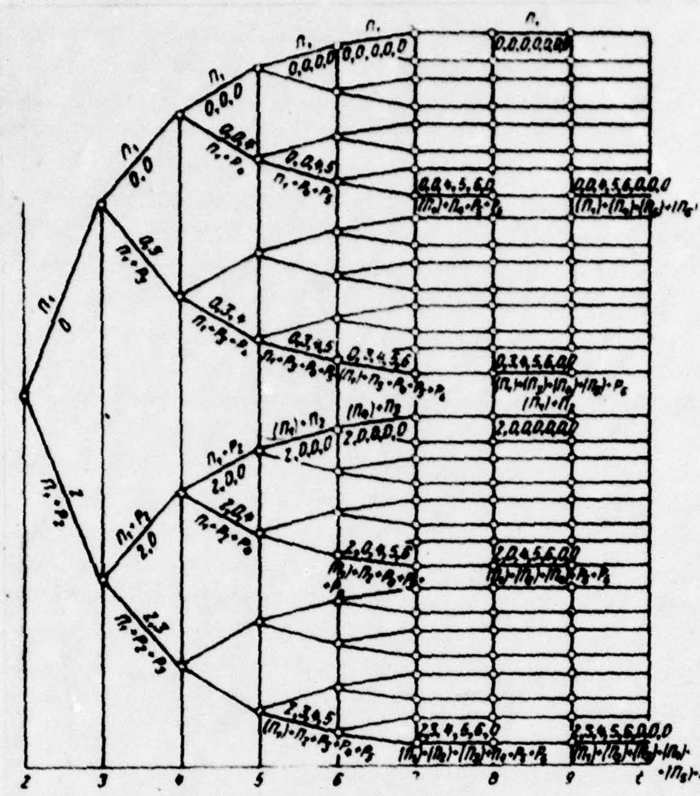

Figure 4.2.1. Possible replacement policies.

Indexing of times of start of development. Time of development is taken as 3 years. (Π_i indicates stopping production of models of i-type).

If $T-\tau_p=10$, then $M=1024$; if $T-\tau_p=20$, then $M\approx10^6$.

As analysis of the algorithm of the problem showed, the number of variants, equal to 10^6, is relatively quickly examined in a modern computer (in 2 hours in a "Ural-2" computer). The basic means of calculation can be the complete sorting method, which is not connected with any assumptions and can be used as the basis for proving any other method.

All possible replacement policies, written in the form of a binary number, are easily formulated in the computer. A binary number is taken with the number of places equal to the number of years in the period under consideration. This number is equal to zero and is written in the form of the corresponding number of zeros. In sequence each space is increased by one until a number is obtained which consists only of ones.

Then summary expenditures must be calculated for each policy, observing the condition of performing all tasks. In the calculation we assume that at the beginning of the year in question the available number of suitable devices corresponds to the required number. Expenditures preceeding this time are not considered and not included in summary expenditures. Then we calculate for each year, in sequence determining the initial number of working devices of each type, the necessary volume of production of the type which is now being produced, summary yearly expenditures, expenditures reduced to the initial moment of time, and expenditures for the entire period in question.

These expenditures are registered and stored in the machine memory until in a following consideration of replacement policy, summary expenditures are less than those registered in the memory. Then the obtained lower expenditures and corresponding replacement policies are registered. Calculation is continued until all variants are completely reviewed.

— increase in the required number of devices of the first type is calculated according to formula

$$\Delta N(t_{p_1};\ t_i) = N(t_{p_1};\ t_{i-1}) - N(t_{p_1};\ t_i); \tag{2}$$

— the number of devices of the first type which have failed, —

$$\Delta N_{в}(t_{p_1};\ t_i) = \sum_{t_{n_j}=0}^{t_{n_j}=t_{i-1}} \Delta N_{n}(t_{p_1};\ t_j)\,[\nu(t_{p_1};\ t_{i-1} - t_{nj}) - \nu(t_{p_1};\ t_i - t_{n_j})], \tag{3}$$

where ΔN_{n} — number of devices produced;

— number of devices produced per year according to formula

$$\Delta N_{n}(t_{p_1};\ t_i) = \Delta N(t_{p_1};\ t_i) + \Delta N_{в}(t_{p_1};\ t_i); \tag{4}$$

— available number of devices according to formula

$$N_{x}(t_{p_1};\ t_i) = N(t_{p_1};\ t_i) \tag{5}$$

until start of production of devices of the next type.

After production begins of the second type of devices

$$N_{\text{x}}(t_{p_1};\ t_i)=N_{\text{x}}(t_{p_1};\ t_{i-1})-\Delta N_{\text{B}}(t_{p_1};\ t_i). \tag{6}$$

Portion of problems solved by devices of the first type

$$P(t_{p_1};\ t_i)=\frac{N_{\text{x}}(t_{p_1};\ t_i)}{N(t_{p_1};\ t_i)}. \tag{7}$$

The required number of devices of the second type is determined according to formula

$$N'(t_{p_2};\ t_i)=N(t_{p_1};\ t_i)\,[1-P(t_{p_1};\ t_i)]. \tag{8}$$

Then the required number of devices of the second type is calculated analogously to devices of the first type, based on N'. Calculation is conducted in the same way for devices of succeeding types, taking into account that instead of formula (8) usually

$$N'(t_{p_i};\ t_i)=N(t_{p_1};\ t_i)\left[1-\sum_{j=1}^{j=t_{p(i-1)}}P(t_{p_j};\ t_i)\right]. \tag{9}$$

Summary expenditures C_{x} are calculated on the basis of $\Delta N_{\text{B}}(t_{p_i};\ t_i)$ and $\Delta N_{\text{x}}(t_{p_i};\ t_i)$.

Example 4.2.1. Let the requirement for a certain kind of automobiles be the following function of current time t and time of completion of development of a given i-type automobile t_{p_i} be:

$$N=1000\exp(0.2083t-0.139\,t_{p_i}). \tag{10}$$

We consider time T = 15 years. Time of development of the automobile τ_p = 5 years. The function of survival has a stepped aspect: k_i = 3 years after production the car fails. The cost of production of automobiles developed in later years increases according to law

$$C=C_0\exp(0.0693t_{p_i});\ \ \mu=0.7.$$

the coefficient of storage cost $k_x = 0{,}5$, the coefficient of development cost $k_p = 63$, $a = 1$.

Optimal times of completing development of new types of automobiles must be selected in which summary expenditures are minimized.

As an example we present the following replacement policy Γ = 000001000000000, i.e. production of a new type begins one time in the sixth year, and for it we calculate C_Σ, which is shown in Table 4.2.1.

We calculate N according to formula (10); ΔN — according to formula (2); ΔN — according to formula (3); ΔN_n — according to formula (4); N_x — according to formula (5) and (6); P — according to formula (7); N' — according to formula (8). Results of calculations are given in Table 4.2.1.

Summary expenditures are calculated according to formula

$$C_\Sigma = C(0)\left\{\left[\sum_t \Delta N_n(0;\ t)\right]^{0,7} + k_x \sum_t N_x(0;\ t)\right\} +$$
$$+ C(5)\left\{\left[\sum_t \Delta N_n(5;\ t)\right]^{0,7} + k_x \sum N_x(5;\ t) + k_p\right\} =$$
$$= 0{,}001\,\{4350^{0,7} + 0{,}5\cdot 13050\} + 0{,}001\exp(0{,}0693\cdot 5)\,\{20684^{0,7} +$$
$$+ 0{,}5\cdot 50812 + 63\} = 44{,}359.$$

These calculations must be conducted for all 2048 possible variants of replacement policy.

Results of calculations, conducted with computers, are given in Table 4.2.2., where C_0 is taken as equal to 0.001.

From the Table it can be seen that the maximal summary expenditures are almost two times greater than minimal expenditures. This means that only by proper planning of new

TABLE 4.2.1

ORDER OF CALCULATING SUMMARY EXPENDITURES

[table reads left to right]

year t	For devices of 1st type					
	$N(0;t)$	$\Delta N(0;t)$	$\Delta N_{\text{в}}(0;t)$	$\Delta N_{\text{п}}(0;t)$	$N_{\text{х}}(0;t)$	$P(0;t)$
0	1000			→ 1000	1000	1,00
1	1234	234	—	234	1234	1,00
2	1520	286	—	286	1520	1,00
3	1862	342	1000	1342	1862	1,00
4	2300	438	234	672	2300	1,00
5	2830	530	286	816	2830	1,00
6	3490		1342		1488	0,426
7	4306		672		816	0,189
8	5294		816		0	0
9						
10						
11						
12						
13						
14						
15						
Σ				4350	13050	

Year, t	For devices of 2nd type					
	$N(5;t)$	$N'(5;t)$	$\Delta N(5;t)$	$\Delta N_{\text{в}}(5;t)$	$\Delta N_{\text{п}}(5;t)$	$\Delta N_{\text{х}}(5;t)$
0						
1						
2						
3						
4						
5	Start of production of technical devices of 2nd type / Начало производства технического устройства 2-го типа					
6	1742	1000	1000	—	1000	1000
7	2145	1740	740	—	740	1740
8	2639	2639	899	—	899	2639
	Failure of all devices of 1st type / Выход из строя всех устройств 1-го типа					
9	3254	3254	615	1000	1615	3254
10	4010	4010	756	740	1496	4010
11	4936	4936	926	899	1825	4036
12	6080	6080	1144	1615	2759	6080
13	7486	7486	1406	1496	2902	7486
14	9217	9217	1731	1825	3556	9217
15	11350	11350	2133	1759	3892	11350
Σ					20684	50812

TABLE 4.2.2

Number of replacements	Number of possible variants of replacement policy	Minimum summary expenditures	Formula of optimal replacement policy	Maximal summary expenditures
0	1	50,8278	000000000000000	50,8278
1	11	41,7024	000100000000000	$\boxed{71,6523}$
2	55	$\boxed{39,5067}$	010000010000000	63,4184
3	166	40,00771	100100010000000	57,6639
4	331	40,6569	110100010000000	53,9926
5	460	41,6181	111100010000000	51,5088
6	460	42,5879	111110010000000	50,6633
7	331	43,9583	111111010000000	50,3516
8	166	45,5911	111111110000000	50,3242
9	55	47,5081	111111111000000	50,8583
10	11	49,4235	111111111100000	50,7387
11	1	51,3658	111111111110000	51,3659

developments can a significant reduction of expenditures be achieved (in all cases demands are completely satisfied).

4.3. Use of the Random Search Method

This method does not guarantee finding an accurate solution, but in those cases when we do not have sufficient machine time available for organizing complete sorting, it can be extremely useful, as it provides an approximate solution.

In this problem blind random search is organized very simply. In fact, assuming the equally probable appearance of zero or one in each place of the number describing replacement policy Γ, independent statistical tests can be performed in each place and, comparing the numerical value of the function from the obtained random number Γ, it is possible to judge the success of the step. Having stored the best result, it is possible with a sufficient number of samples (steps) to find a replacement policy close to optimum.

From search theory we know that blind random search is much less effective than sequential random search.

Studies [53] have shown that the algorithm with the "best initial sample" was more effective. The essence of this algorithm is as follows.

At the start of search, several steps are taken, organized according to the principle of blind random search. The step is selected at which the least value of the functional was obtained (best sample), and subsequent search is carried out sequentially from the best sample.

The general form of the algorithm of sequential random search can be described in the form of recurrent expression

$$P_n^j = P_*^j + \xi_n^j \Lambda, \tag{1}$$

where P_n^j — probability of the appearance of "1" in j-place of the graph at n-step; P_*^j — probability of the appearance of "1" at the next successful step; ξ_n^j — random number, equally probably distributed in segment $- 1 + 1$; Λ — scale of step.

To determine Γ, in each space P_n^j is compared with 0.5 and if $P_n^j \geq 0.5$, then a 1 is entered in this place, otherwise — 0. Thus the number of Γ is formulated.

The random step concludes with proof of its success, for which the numerical value of functional $C_z(\Gamma_n)$ is determined and it is compared with the minimum value, obtained in all preceeding steps. The step is assumed to be successful if the following inequality is fulfilled

$$C(\Gamma_n) < C(\Gamma_*), \tag{2}$$

where $C(\Gamma_*) = \min C(\Gamma_\varphi),\ \varphi = 1, 2, \ldots, n - 1$.

In case of a successful step, the new value $P_*^j = P_n^j$ is assumed at this successful step and the following $(n + 1)$ step is conducted based on (1), but with the new value of P_*^j. With an unsuccessful step, no change is produced in (1).

> <u>Example 4.3.1</u>. Solve the problem presented in example 4.2.1 by the random search method.
>
> First we use the blind random search method.
>
> From the table of random numbers, distributed according to the law of even probability [12] in the interval from 0 to 1, we

take a total of 11 numbers and compare them with 0.5. If the random number is less than 0.5, then in the corresponding place of replacement policy Γ we write 0, otherwise — 1. The last 4 numbers in our example are zero.

Thus, for the first sample we obtain 0.57705, 0.71618, 0.73710, 0.70131, 0.16961, 0.53324, 0.43166, 0.26275, 0.05926, 0.66289, 0.35483.

From that, the formula of the replacement policy for the first sample is $\Gamma_1 = 111101000100000$.

Based on this value, we calculate C_{Σ_1} according to the method described in example 4.2.1 and determine Γ_2, C_{Σ_2} , etc., storing Γ* corresponding to the least value of C_Σ.

As this example was solved by an accurate method, it was *possible to evaluate the number of samples* in random search as a function of the deviation of the found value of C^*_Σ with actually optimal $C_{\Sigma\,opt}$:

$$\delta = \frac{C^*_\Sigma - C_{\Sigma\,opt}}{C_{\Sigma\,opt}}.$$

Corresponding data are given in Fig. 4.3.1. There are shown expenditures of samples necessary in complete description and in sequential random search, the algorithm of which is described above. In this example it was found that about 40% of machine time should be spent "for initial samples."

From the illustration it is evident that sequential random search is much more effective than blind search. If accuracy under 5% is sufficient, then the method of random search can reduce machine time by one order of magnitude using blind search and two orders of magnitude using sequential search.

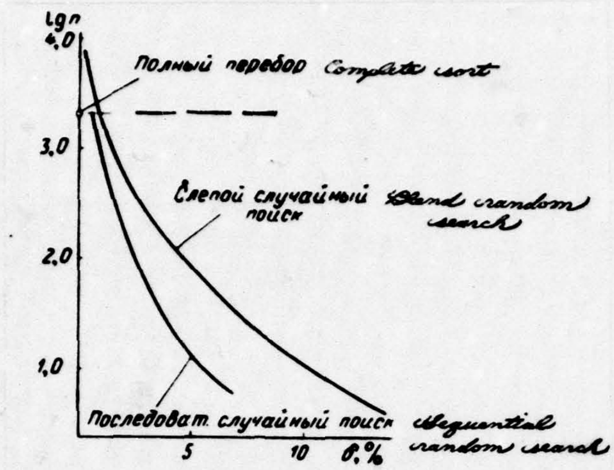

Figure 4.3.1. Comparative evaluation
of search methods

4.4. Solving Problems using the Discrete Principle of Maximization

Let us consider the problem described in § 4.1, having made the following additional assumptions.

First, let us consider the function of survival $\nu(t_{p_i};\ t-t_n)$ to be stepped, i.e. assume that

at $t-t_n<k_i$ $\nu(t_{p_i};\ t-t_n)=1;$

at $t-t_n>k_i$ $\nu(t_{p_i};\ t-t_n)=0,$

where k_i – survival time of devices of i-type.

Second, we shall not reduce expendutires to a single moment of time.

Third, we shall consider the cost of production and exploitation as proportional to the amount of produced and exploited technical devices ($\mu = 1$).

The process of replenishing supplies in j-year consists of increasing the number of technical devices in operation by the production of new ones $\Delta N_n(t_{p_i};\ t_j)$ and reducing the number of technical devices as a result of their failure $\Delta N_n(t_{p_i};\ t_j-k_i)$ (taking into account the above assumptions).

The quality of serviceable devices of the i-th type suitable for use in the j-th year can be computed from the formula

$$N_{\text{х}}(t_{p_i};\ t_j) = N_{\text{х}}(t_{p_i};\ t_{j-1}) + \\ + \Delta N_{\text{п}}(t_{p_i};\ t_j) - \Delta N_{\text{п}}(t_{p_i};\ t_j - k_i), \qquad (1) \\ i = 1, 2, \ldots, k.$$

Equation (1) is an equation with a delayed argument, in conformity with which optimization methods are poorly developed. Therefore, we introduce those conversions of variables which will make it possible to reduce the descriptions of the process characterized by equation (1) to a system of equations without a lag.

We introduce a new variable, which is the total number of devices of i-type produced in j-year,

$$x^j_{i_1} = \Delta N_{\blacksquare}(t_{p_i};\ t_j), \tag{2}$$

and a new equation

$$\bullet^j_{i_1} = \Delta N_{\blacksquare}(t_{p_i};\ t_j) - \Delta N_{\blacksquare}(t_{p_i};\ t_{j-1}), \tag{3}$$

which is the increment of production in j-year in comparison with (j-1) year.

Then from (2) and (3) follows:

$$x^j_{i_1} = x^{j-1}_{i_1} + \bullet^j_{i_1}. \tag{4}$$

We introduce a new variable

$$x^j_{i_0} = \bullet^j_{i_1} \tag{5}$$

and a new equation

$$\bullet^j_{i_0} = \bullet^j_{i_1} - \bullet^{j-1}_{i_1}. \tag{6}$$

Substituting $\bullet^j_{i_1}$ from (6), taking into account (5) in (4), we obtain

$$x^j_{i_1} = x^{j-1}_{i_1} + x^{j-1}_{i_0} + \bullet^j_{i_0}$$

and from (5) and (6)

$$x^j_{i_0} = x^{j-1}_{i_0} + \bullet^j_{i_0}.$$

The system of analogous conversions is conducted sequentially k_i times (where k_i — time when technical devices are suitable, in units of the time reference of the process).

As a result of these conversions, equation (1) can be represented in the form of the following system of equations with no delay:

$$
\begin{cases}
N_x(t_{p_i};\ t_j) = N_x(t_{p_i};\ t_{j-1}) + \omega^j_{i_{ni}}\ ; \\
x^j_{i_1} = x^{j-1}_{i_1} + x^{j-1}_{i_2} + \ldots + x^{j-1}_{i_{ni}} + \omega^j_{i_{ni}}\ ; \\
x^j_{i_2} = x^{j-1}_{i_2} + x^{j-1}_{i_3} + \ldots + x^{j-1}_{i_{ni}} + \omega^j_{i_{ni}}\ ; \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
x^j_{i_{ni}} = x^{j-1}_{i_{ni}} + \omega^j_{i_{ni}}\ .
\end{cases}
\tag{7}
$$

In this system, for each step of j are stored data on the number of technical devices by their ages (times from the moment of manufacture), which are determined by x_{i_l} — the number of devices of i-type produced in $(j - l + 1)$ year.

Then the following dependence can be used to calculate devices produced by j-year:

$$
\begin{aligned}
N_x(t_{p_i};\ t_j) = N_x(t_{p_i};\ t_{j-1}) + \\
+ x^{j-1}_{i_1} + x^{j-1}_{i_2} + \ldots + x^{j-1}_{i_{ni}} + \omega^j_{i_{ni}}\ .
\end{aligned}
\tag{8}
$$

Physically the new control $\omega^j_{i_{ni}}$ is the increase in the number of technical devices of i-type ready for use in j-year.

The effectiveness of the technical device is determined by the year development was completed and can be written in the form of equation

$$
Э^j_i = F^j_i(Э^{j-1}_i;\ t_{p_i};\ x^j_i),
\tag{9}
$$

where $Э^j_i$ — the index of effectiveness of i-device in j-year; t_{p_i} — time development of the technical device was completed; $x^j_i \in \{0;1\}$ — index of the start of production of devices ($x^j_i = 0$ — device not produced, $x^j_i = 1$ — start of production of device).

In a particular case, instead of (9) we can write

$$
Э^j_i = Э^{j-1}_i + a_i (1 - x^j_i),
\tag{10}
$$

where a_i — the increment of effectiveness in one step.

For variable x^j_i , determining the start of production of the device, the equation of conversion can be written as follows:

$$
x^j_i = x^{j-1}_i + \theta^j_i,
$$

where

$$x_i^0 = 0; \quad x_i^N = 1; \quad \theta_i^j \in \{0; 1\}. \tag{11}$$

The general productivity of technical devices is considered as given and can be calculated in linear approximation according to a dependence of the type

$$\Pi^j = \sum_i F_i^n (\Theta_i^{j-1}, x_i^{j-1}, t_{p_i}) [N_x (t_{p_i}; t_{j-1}) + \omega_{ik_i}^j]. \tag{12}$$

The equation for the cost of technical devices, taking into account expenditures for their development, production and exploitation, can be represented by an expression of the kind

$$C^j = C^{j-1} + \sum_i \Big\{ C_{p_i} \theta_i^j + C_{o_i} \Big[N_n (t_{p_i}; t_{j-1}) + \sum_{j=1}^{k_i} x_{ij}^{j-1} + \omega_{ik_i}^j \Big] + C_{x_i} [N_x (t_{p_i}; t_{j-1}) + \omega_{ik_i}^j] \Big\}. \tag{13}$$

Thus, the process of replacement of technical devices is described by equations (7) and (8), determining the dynamics of production and exploitation; by equations (9)-(11), characterizing the effectiveness of the device as a function of time of completion of its development; with equations (12) and (13), describing the productivity of operational technical devices and complete material expenditures to ensure their function.

Boundary conditions have the following appearance:

$$\begin{gathered} N_x (t_{p_i}; 0) = N_{x_i} (0), \; i = 1, 2, \ldots, K; \\ x_{ij}^0 = x_{0_{ij}}, \; i = 1, 2, \ldots, K; \; j = 1, 2, \ldots, K; \\ x_i^0 = 0, \; x_i^T = 1. \end{gathered} \tag{14}$$

In this problem, $N_x, x_{ij}, N_n, \Theta_i, x_i$ are phase coordinates, and θ_i and ω_{ik_i} — controls, i.e. the process can be controlled by selecting the time of completion of development of new kinds of technical devices and the volume of their production.

As the volume of production at each step is given, equation (12) is used to determine one of the controls, for example, the volume of production of technical devices in operation at the start of the period in question (i = 1).

Let for them

$$\mathit{Э}_1 = 1, \; x_i^{j-1} = 1.$$

Then from (12), taking into account (10), we obtain

$$\begin{aligned} \omega_{1k_1}^j &= \Pi^j - N_x(t_{p_1};\, t_{j-1}) - \\ &- \sum_{i=2}^{k} [\mathit{Э}_i^{j-1} + a_i(1 - x_i^{j-1})]\, [N_x(t_{p_i};\, t_{j-1}) + \omega_{i_{ki}}^j]. \end{aligned} \tag{15}$$

In accordance with the procedure of solving the problem of optimization using the discrete principle of maximization, we write a Hamilton function and substitute in it ω_{1k_1} instead of (15):

$$\begin{aligned} H^j = &\Big\{\Pi^j - N_x(t_{p_1};\, t_{j-1}) - \sum_{i=2}^{K} [\mathit{Э}_i^{j-1} + \\ &+ a_i(1 - x_i^{j-1})]\, [N_x(t_{p_i};\, t_{j-1}) + \omega_{i_{ki}}^j]\Big\} \times \\ &\times \Big(\psi_{N_{x_1}}^j + \sum_{\mu=1}^{k_1} \psi_{x_{1\mu}}^j + \psi_{N_{\Pi_1}}^j + C_{o_1} + C_{x_1}\Big) + \\ &+ \psi_{N_{x_1}}^j N_x(t_{p_1};\, t_{j-1}) + \sum_{\mu=1}^{k_1} \psi_{x_{1\mu}}^j x_{1\mu}^{j-1} + \\ &+ \psi_{N_{\Pi_1}}^j \Big[N_\Pi(t_{p_1};\, t_{j-1}) + \sum_{\mu=1}^{k_1} x_{1j}\Big] + \\ &+ C_{o_1}\Big[N_\Pi(t_{p_1};\, t_{j-1}) + \sum_{\mu=1}^{k_1} x_{1\mu}^{j-1}\Big] + C_{x_1} N_x(t_{p_1};\, t_{j-1}) + \\ &+ \sum_{i=2}^{K} \Big\{\psi_{N_{xi}}^j \Big[N_x(t_{p_i};\, t_{j-1}) + \omega_{i_{k_i}}^j\Big] + \\ &+ \sum_{\mu=1}^{k_i} \psi_{x_{i\mu}}^j (x_{i\mu}^{j-1} + \omega_{i_{k_i}}^j) + \psi_{N_{\Pi_i}}^j \Big[N_\Pi(t_{p_i};\, t_{j-1}) + \\ &+ \sum_{j=1}^{k_1} x_{i\mu}^{j-1} + \omega_{i_{k_i}}^j\Big] + \psi_{\mathit{Э}_i}^j \Big[\mathit{Э}_i^{j-1} + a_i(1 - x_i^{j-1})\Big] + \\ &+ \psi_{x_i}^j (x_i^{j-1} + \theta_i^j) + C_{p_i} \theta_i^j + C_{o_i}\Big[N_\Pi(t_{p_i};\, t_{j-1}) + \sum_{\mu=1}^{k_1} x_{i\mu}^{j-1} + \\ &+ \omega_{i_{k_i}}^j\Big] + C_{x_i} [N_x(t_{p_i};\, t_{j-1}) + \omega_{i_{ki}}^j]\Big\} + C^{j-1}. \end{aligned} \tag{16}$$

Conjugate vector ψ is determined by a system of equations

$$\psi = \frac{\partial H^n}{\partial x^{n-1}}.$$

On this basis, we write equations for a conjugate system:

$$\psi^{j-1}_{N_{x_i}} = \frac{\partial H^j}{\partial N_x(t_{p_i}; t_{j-1})} = -\Big(\psi^j_{N_{x_i}} + \sum_{\mu=1}^{k_i} \psi^j_{x_{1\mu}} + \psi_{N_{n_i}} + C_{o_i} + C_{x_i}\Big)\left[Э^{j-1}_i + a_i(1 - x^{j-1}_i)\right] + \psi^j_{N_{x_i}} + C_{x_i}; \tag{17}$$

$$\psi^{j-1}_{N_{n_i}} = \frac{\partial H^j}{\partial N_n(t_{p_i}; t_{j-1})} = \psi^j_{N_{n_i}} + C_{o_i}; \tag{18}$$

$$\psi^{j-1}_{x_{i\mu}} = \frac{\partial H^j}{\partial H^{j-1}_{i\mu}} = \sum_{\mu=1}^{k_i} \psi^j_{x_{i\mu}} + \psi^j_{N_{n_i}} + C_{o_i}; \tag{19}$$

$$\psi^{j-1}_{Э_i} = \frac{\partial H^j}{\partial Э^{j-1}_i} = -\left(\psi^j_{N_{x_i}} + \sum_{\mu=1}^{k_i} \psi^j_{x_{1\mu}} + \psi^j_{N_{n_i}} + C_{o_i} + C_{x_i}\right)\left[N_n(t_{p_i}; t_{j-1}) + \omega^j_{i_{ki}}\right] + \psi^j_{Э_i}; \tag{20}$$

$$\psi^{j-1}_{x_i} = \frac{\partial H^j}{\partial x^{j-1}_i} = a_i\left\{\left[N_x(t_{p_i}; t_{j-1}) + \omega^j_{i_{ki}}\right]\left(\psi^j_{N_{x_i}} + \sum_{\mu=1}^{k_i} \psi^j_{x_{1\mu}} + \psi^j_{N_{n_i}} + C_{o_i} + C_{x_i}\right) - \psi^j_{Э_i}\right\}. \tag{21}$$

In all of these equations $i \neq 1$.

Equations for technical devices of the first type will have the following appearance:

$$\psi^{j-1}_{N_{x_1}} = \frac{\partial H^j}{\partial N_{x_1}(t_{p_1}; t_{j-1})} = -\sum_{\mu=1}^{k_1} \psi^j_{x_{1\mu}} - \psi^j_{N_{n_1}} - C_{o_1}; \tag{22}$$

$$\psi^{j-1}_{N_{n_1}} = \frac{\partial H^j}{\partial N_n(t_{p_1}; t_{j-1})} = \psi^j_{N_{n_1}} + C_{o_1}; \tag{23}$$

$$\psi^{j-1}_{x_{1\mu}} = \frac{\partial H^j}{\partial x^{j-1}_{1\mu}} = \sum_{\mu=1}^{k_1} \psi^j_{x_{1\mu}} + \psi^j_{N_{n_1}} + C_{o_1}. \tag{24}$$

From (22) and (24)

$$\psi^{j-1}_{N_{x_1}} = -\psi^{j-1}_{x_{1\mu}}. \tag{25}$$

Boundary conditions for generalized impulses in accordance with the method of solving the problem [49] after conversion can be represented in the form

$$\psi^T_{N_{x_i}} = \frac{\partial C^T}{\partial N_x(t_{p_i}; t_{T-1})} = C_{x_i},\ i = 1, 2, \ldots, K; \tag{26}$$

$$\psi^T_{N_{n_i}} = \frac{\partial C^T}{\partial N_n(t_{p_i}; t_{T-1})} = C_{o_i},\ i = 1, 2, \ldots, K; \tag{27}$$

$$\psi^T_{x_{i\mu}} = \frac{\partial C^T}{\partial x^{T-1}_{i\mu}} = C_{o_i},\ i = 1, 2, \ldots, K; \tag{28}$$

$$\psi^T_{\ni_i} = \frac{\partial C^T}{\partial \ni^{T-1}_i} = 0,\quad i = 1, 2, \ldots, K. \tag{29}$$

Using (23), with $j = T - 1$, etc., we obtain

$$\begin{aligned} \psi^{T-1}_{N_{n_i}} &= \psi^T_{N_{n_i}} + C_{o_i} = 2C_{o_i}; \\ \psi^{T-2}_{N_{n_i}} &= \psi^{T-1}_{N_{n_i}} + C_{o_i} = 3C_{o_i}; \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi^{T-m}_{N_{n_i}} &= (m+1)\,C_{o_i}. \end{aligned} \tag{30}$$

Calculating $\psi_{x_{i\mu}}$ with $j = T;\ j = T - 1$, etc., using (24) and calculating (28), by induction we can come to a general dependence of the type

$$\psi^{T-m}_{x_{i\mu}} = -\psi^{T-m}_{N_{x_i}} = C_{o_i}\,(k^m_i + 2k^{m-1}_i + \ldots + mk_i + m + 1). \tag{31}$$

Analogously from equations (18) and (27) we obtain

$$\psi^{T-m}_{N_{n_i}} = (m+1)\,C_{o_i}. \tag{32}$$

and from equations (19) and (28) –

$$\psi^{T-m}_{x_{i\mu}} = C_{o_i}\,(k^m_i + 2k^{m-1}_i + 3k^{m-2}_i + \ldots + mk_i + m + 1). \tag{33}$$

For integration of equation (17) we must have information on the replacement policy, which is expressed in x^{j-1}_i.

In connection with this, equation (17) can be integrated, taking into account two assumptions: production of the technical device has not yet begun $(x^{j-1}_i = 0)$ and production has begun $(x^{j-1}_i = 1)$.

In the first case, assuming that in m steps until the end of the process $x_i^{j-1}=0;\ \mathcal{E}_i^{T-m}=1+a_i\,(T-m)$, and assuming $x_i^{j-1}=0$, we obtain

$$\psi_{N_{x_i}}^{T-m}=-[1+a_i(T-m_i)][C_{0_i}(k_i^{m+1}+2k_i^m+ +3k_i^{m-1}+\ldots+(m+1)k_i^3+mk_i^2+mk_i+m)+ +mC_{x_i}]+(m+1)C_{x_i}. \tag{34}$$

In the second case (production has begun) effectiveness becomes a constant and the general appearance of dependence (34) is preserved, only term $a_i(T-m)$ is replaced by term $a_i(T-m_i)$, where $t_{pi}=(T-m_i)$ — year of the start of production of i-technical device.

From consideration of the Hamilton function (16), it can be noted that equations enter it linearly. The principle of maximization requires finding minimum H^j according to controls with satisfaction of boundary conditions.

We extract from (16) terms containing controls $\omega_{ik_i}^n$:

$$H^{j*}=\{-[\mathcal{E}_i^{j-1}+a_i\,(1-x_i^{j-1})](\psi_{N_{x_1}}^n+ +\sum_{j=1}^{k_1}\psi_{x_{1j}}^n+\psi_{N_{n_1}}^j+C_{0_1}+C_{x_1})+\psi_{N_{x_1}}^j+ +\sum_{\mu=1}^{k_i}\psi_{x_{i\mu}}^j+\psi_{N_{n_i}}^j+C_{0_i}+C_{x_i}\}\omega_{ik_i}=A\omega_{ik_i}^j. \tag{35}$$

Function A plays the role of the function of shifting in the theory of control of systems with linear controls. To ensure minimum H^{j*} according to $\omega_{ik_i}^j$, the sign of function A must be analyzed, and in this case

$$\omega_{ik_i}^j=\begin{cases}0 & \text{at } A>0,\\ 1 & \text{at } A<0.\end{cases} \tag{36}$$

Of interest for analysis is the time when the new type of technical devices goes into production. In this case $x_i^{j-1}=1$.

From dependences (17) and (35), it can be noted that

$$A=\psi_{N_{x_i}}^{j-1}+\sum_{\mu=1}^{i}\psi_{x_{i\mu}}^j+\psi_{N_{n_i}}^j+C_{0_i}. \tag{37}$$

Then, taking into account equation (34), ratio (37) can be rewritten in the form

$$A = -[1 + a_i(T - m)]\left[k_i^m + 2k_i^{m-1} + \dots + (m-1)k_i^2 + \right.$$
$$\left. + (m-1)k_i + (m-1)\left(1 + \frac{C_{x_i}}{C_{o_i}}\right) + m\frac{C_{x_i}}{C_{o_i}}\right] +$$
$$+ C_{x_i}^{m+1} + 2k_i^m + \dots + (m-1)k_i^3 + mk_i^2 + \qquad (38)$$
$$+ mk_i + k_i + m + 2,$$

where powers of k_i are greater than zero. The shift of the sign in expression (38) must correspond to the start of production of technical devices of i-type.

For the case when the cost of development is not calculated, this solution will be final. Calculation of the cost of development is connected with additional calculation difficulties, which consist of the necessity of simultaneously guaranteeing minimum H^j, not only according to $\omega_{i_{k_i}}^j$, i.e. volume of production, but also according to the time of completion of development of technical devices of i-type θ_i^j. These controls also enter the Hamilton function linearly:

$$H^{j**} = (\psi_{x_i}^j + C_{p_i})\theta_i^j = B\theta_i^j. \qquad (39)$$

B is the function of switching for θ_i^j, and the shift of the sign of this function will correspond to the time of completion of development of a device of i-type:

$$\theta_i^j = \begin{cases} 0 & \text{at } B > 0; \\ 1 & \text{at } B < 0. \end{cases} \qquad (40)$$

However, here it is necessary to integrate equation (21), which is connected with the necessity of also integrating equation (20). Both these equations contain controls $\omega_{i_{k_i}}^j$, in connection with which solution of the problem concerning replacement first requires obtaining data on the start of production in accordance with dependence (37) and the function of switching (36), and then correcting the solution according to controls θ_i^j.

Thus, with the above assumptions, an analytical solution can be obtained.

Example 4.4.1. Assuming the cost of development to be zero (which corresponds to a case of very large volume of production)

and taking the cost of exploitation $C_{x_i} < C_0$ the cost of production, determine the optimal ratio of effectiveness of technical devices at the time of replacement if survival time of the device k_i is 1, 2 and 5 years.

This problem can be solved using dependence (38) and conditions of switching (36).

In expression (38), on the basis of (10), term $[1 + a_i(T-m)] = \frac{Э_{эсм}}{Э_0}$ where $Э_0$ is taken as 1. Then expression (38) takes the form

$$A = -\frac{Э_{эсм}}{Э_0}\left[k_i^m + 2k_i^{m-1} + \ldots + (m-1)k_i^2 + (m-1)k_i + (m-1)\times \right.$$
$$\left. \times\left(1 + \frac{C_{x_i}}{C_{0_i}}\right) + m\frac{C_{x_i}}{C_{0_i}}\right] + C_{x_i}^{m+1} + 2k_i^m + \ldots + (m-1)k_i^3 +$$
$$+ mk_i^2 + mk_i + k_i + m + 2.$$

In accordance with (36), replacement will occur when A changes sign, i.e. passes through zero. Equating A to zero, we obtain

$$\frac{Э_{эсм}}{Э_0} = \frac{C_{x_i}^{m+1} + 2k_i^m + \ldots + (m-1)k_i^3 + mk_i^2 +}{k_i^m + 2k_i^{m-1} + \ldots + (m-1)k_i^2 +} \rightarrow$$
$$\leftarrow \frac{+ mk_i + k_i + m + 2}{+ (m-1)k_i + (m-1)\left(1 + \frac{C_{x_i}}{C_{0_i}}\right) + m\frac{C_{x_i}}{C_{0_i}}}.$$

The limiting case will be $m = 1$, at which replacement occurs a year before the end of period T. In this case, taking into account $C_x = 0$, we write the preceeding equation as follows:

$$\frac{Э_{эсм}}{Э_0} = \frac{k_i^2 + 2k_i + 3}{k_i + 2} = k_i + \frac{3}{k_i + 2}.$$

Substituting $k_i = 1$, we obtain

$$\frac{Э_{эсм}}{Э_0} = 1 + \frac{3}{1+2} = 2;$$

with $k_i = 2$

$$\frac{Э_{эсм}}{Э_0} = 2 + \frac{3}{2+2} = 2{,}75;$$

with $k_i = 5$

$$\frac{Э_{эсм}}{Э_0} = 5 + \frac{3}{5+2} = 5{,}4.$$

From these results it is clear that replacement of technical devices with long survival time becomes feasible with high gains in effectiveness in comparison with devices of the old type, i.e. the obsolescence of technical devices with long survival time occurs more slowly than that of technical devices with short survival time.

4.5. Very Simple Cases of Solving Generalized Problems of Replacement

Let us consider the problem formulated in § 4.1, with simplified assumptions for particular cases.

First case. Let $\nu(t_{p_i};\ t-t_n)=1$, i.e. we ignore the failure of technical devices in proportion to storage or exploitation.

Let demands be described by a formula of the following type:

$$N(t_{p_i};\ t)=\frac{N_0 t}{t_{p_i}} \tag{1}$$

within limits from t_0 to T.

Let

$$C_\Sigma=\sum_{i=1}^{K}[C_0 N_i+C_p], \tag{2}$$

i.e. it is assumed that production cost of devices does not depend on the size of the lot or the time of their development and development cost of devices does not depend on the time of completion of development.

Then the following equation can be written for summary cost in the first development of a new technical device

$$C_\Sigma=C_0\frac{N_0 t_1}{t_0}+C_0\frac{N_0(T-t_1)}{t_1}+C_p. \tag{3}$$

To determine the optimal value of t_1, we equalize the derivative of C_Σ by t_1 to zero:

$$\frac{C_0 N_0}{t_0}-\frac{C_0 N_0 T}{t_1^2}=0,$$

TABLE 4.5.1

OPTIMAL VALUES OF K

a	b									
	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
2	—	—	—	—	—	—	—	0,10	0,36	0,81
3	0,10	0,18	0,26	0,30	0,39	0,48	0,60	0,80	1,11	1,82
4	0,37	0,42	0,49	0,57	0,66	0,80	0,97	1,30	1,72	2,40
5	0,60	0,66	0,75	0,85	0,97	1,14	1,38	1,66	2,04	3,10
6	0,77	0,85	0,94	1,07	1,22	1,39	1,61	1,90	2,35	3,70
7	0,94	1,03	1,13	1,25	1,39	1,56	1,79	2,11	2,66	3,94
8	1,08	1,16	1,30	1,37	1,51	1,70	1,96	2,36	2,96	4,30
9	1,21	1,30	1,40	1,53	1,68	1,87	2,10	2,48	3,18	4,86
10	1,29	1,39	1,50	1,63	1,80	2,01	2,28	2,68	3,48	5,00

Example 4.5.1. Let the function of demand for computers be approximated by a dependence of the type

$$N=\frac{200\,t}{t_{p_t}}$$

within limits from $t_{p_t}=5$ to $t_{p_t}=25$; from $t_0'=5$ to $t=25$.

The cost of production of one computer C_0 = 1,000 rubles, cost of development C_ϕ = 100,000 rubles.

Let us consider period T, equal to 20 years.

We must determine optimal times for putting new types of computers into production.

We must take t_0 = 5.

We calculate, using formula (8),

$$a=\frac{20+5}{5}=5$$

and, using formula (9),

$$b = 1 - \frac{100\,000}{1000 \cdot 200} = 0.5.$$

Using the table of optimal values of K, we find that K = 1.14. Using formula (7), we verify C_Σ with K = 1 and K = 2:

$$C_{\Sigma_1} = 1000 \cdot 200 \left[2\sqrt[3]{\frac{25}{5}} - 1 \right] + 100\,000 \cdot 1 = 794\,400;$$

$$C_{\Sigma_2} = 1000 \cdot 200 \left[3\sqrt[3]{\frac{25}{5}} - 2 \right] + 100000 \cdot 2 = 826000.$$

Therefore, the optimal value will be K = 1, i.e. one new type of computer must be developed and put into production in the considered interval of time. According to formula (6) we obtain

$$t_1 = \sqrt{5 \cdot 20} = 10.$$

Taking into account the accepted reference point t_0 = 5, a new type of computer must be put into production every 10-5 = 5 years.

<u>Second case</u>. Let

$$\nu(t_{p_i};\ t - t_n) = \exp\left[-\lambda_i (t - t_n)\right]; \tag{11}$$

$$N(t_{p_i};\ t) = \frac{I(t)}{a(t_{p_i})}. \tag{12}$$

Then the condition of the sufficiency of technical devices can be written as follows:

$$J(t) = N_0 \nu(t_{p_1};\ t) + \int_0^t \frac{\partial N_n(t_{p_1};\ u)}{\partial t} \nu(t_{p_1};\ t - u)\,du$$

$$\text{with } 0 < t < t_n; \tag{13}$$

$$J(t) = N_0 \nu(t_{p_1};\ t) + \int_0^t \frac{\partial N_n(t_{p_1};\ u)}{\partial t} \nu(t_{p_1};\ t - u)\,du +$$

$$+ a(t_{p_2}) \int_{t_n}^{T} \frac{\partial N_n(t_{p_2};\ u)}{\partial t} \nu(t_{p_2};\ t - u)\,du \quad \text{with } t_n < t < T. \tag{14}$$

In this case, for simplicity we assume $a(t_{p_2}) = 1$.

from which

$$t_1 = \sqrt{t_0 T}; \tag{4}$$

$$C_\Sigma = C_0 N_0 \left[2\sqrt{\frac{T}{t_0}} - 1\right] + C_p. \tag{5}$$

The problem in second, third and other developments is solved analogously. If all of new developments are K, then equations (4) and (5) accordingly assume the following form:

$$t_m = \sqrt[K+1]{t_0^{K+1-m} T^m}, \tag{6}$$

where m — number of developments;

$$\frac{C_\Sigma}{C_0 N_0} = (K+1)\sqrt[K+1]{\frac{T}{t_0}} - K\left(1 - \frac{C_p}{C_0 N_0}\right). \tag{7}$$

The optimal value of K can be found from analysis of this equation.

We designate

$$\frac{T}{t_0} = a; \tag{8}$$

$$1 - \frac{C_p}{C_0 N_0} = b. \tag{9}$$

We differentiate equation (7) by K and equalize the derivative to zero:

$$\sqrt[K+1]{a} - \frac{\ln a \sqrt[K+1]{a}}{K+1} - b = 0. \tag{10}$$

Having solved this transcendental equation, we can find the optimal value of K.

Table 4.5.1 is compiled to simplify calculations. In it dashes indicate values of K less than zero, as having no physical meaning.

Thus, the formulated problem is easily solved using Table 4.5.1.

As K can only be a whole number, values taken from the table must be rounded off, up and down, to a whole number and C_Σ calculated with these two values, selecting the one where C_Σ is least.

Equations (13) and (14) are integral Volterra equations of the second order, which can be solved only in the simplest cases. In particular, if

$$N(t_{pi};\ t)=\begin{cases}0 \text{ with } t<0,\\ N_0 \text{ with } t\geq 0,\end{cases} \tag{15}$$

$$v(t)=\begin{cases}0 & \text{with } t<0,\\ \exp[-\lambda_0(t-t_n) & \text{with } t\geq 0\end{cases} \tag{16}$$

The solution is quite simple

$$\frac{\partial N_n(t_{pi};\ t)}{\partial t}=N_0\lambda_0,$$

from which

$$N_n(t_{pi};\ t)=N_0\lambda_0 t.$$

The total number of devices of the first time which must be produced in time T to ensure N_0 operative devices at any moment of time will be equal to

$$N_1=N_0+N_0\lambda_0 T=N_0(1+\lambda_0 T).$$

Summary expenditures in this case will be

$$C_{\Sigma_1}=C_{n_1}N_0(1+\lambda_0 T)+C_{x_1}N_0 T. \tag{17}$$

If at time t_2 development of devices of the second type is completed and all devices of the first type are replaced with new ones, then summary expenditures are determined as follows:

$$C_{\Sigma_2}=C_{n_1}N_0(1+\lambda_0 t_2)+C_{x_1}N_0 t_2+ \\ +\frac{C_{n_2}N_0[1+\lambda_{0_2}(T-t_2)]}{a(t_{p_2})}+\frac{C_{x_2}N_0(T-t_2)}{a(t_{p_2})}+C_p. \tag{18}$$

From this equation it is necessary to determine t_2, minimizing C_Σ.

In the simplest case $\lambda_0=\lambda_{0_2}=\lambda$;

$$a(t_{p_2})=\frac{1}{a-bt_0};\ C_{n_1}=C_{n_2}=C_n;\ C_{x_1}=C_{x_2}=C_x. \tag{19}$$

Then a formula can be obtained to determine t_2.

After substitution of (19) in (18), we obtain:

$$C_{\Sigma_2} = N_0 \{C_{\text{н}}(1 + \lambda t_2) + C_{\text{з}} t_2 + (a - bt_2) C_{\text{н}} [1 + \lambda(T - t_2)] + \\ + (a - bt_2) C_{\text{з}} (T - t_2)\} + C_{\text{р}}.$$

Then

$$\frac{\partial C_{\Sigma_2}}{\partial t_2} = 0 = C_{\text{н}}\lambda + C_{\text{з}} - bC_{\text{н}} [1 + \lambda (T - t_2)] - \\ - (a - bt_2) C_{\text{н}}\lambda - bC_{\text{з}} (T - t_2) - (a - bt_2) C_{\text{з}},$$

from which after conversion

$$t_2 = \frac{a + bT - 1}{2b} + \frac{C_{\text{н}}}{2 (C_{\text{н}}\lambda + C_{\text{з}})}. \tag{20}$$

Example 4.5.2. Let the demand for computers be expressed by function

$$N = N_0(a - bt_{\text{р}}),$$

where N_0 = 1000; a = 1; b = 0.04 1/year;
Let λ = 0.2 1/year; T = 20 years; $C_{\text{р}}$ = 100,000 rubles;
$C_{\text{н}}$ = 1,000 rubles; $C_{\text{з}}$ = 100 rubles.
Then

$$t_2 = \frac{1 + 0{,}04 \cdot 20 - 1}{2 \cdot 0{,}04} + \frac{1000}{2 (1000 \cdot 0{,}2 + 100)} = 11{,}67 \text{ years.}$$

We now compute C_{Σ_1} and C_{Σ_2} according to (17) and (18)

$$C_{\Sigma_1} = 1000 \cdot 1000 (1 + 0{,}2 \cdot 20) + 100 \cdot 1000 \cdot 20 = 7\,000\,000 \text{ rubles.}$$

$$C_{\Sigma_2} = 1000 \{1000 (1 + 0{,}2 \cdot 11{,}67) + 100 \cdot 11{,}67 + \\ + (1 - 0{,}04 \cdot 11{,}67) \cdot 1\,000 [1 + 0{,}2 (20 - 11{,}67)] + \\ + (1 - 0{,}04 \cdot 11{,}67) \cdot 100 (20 - 11{,}67)\} + 100\,000 = 6\,467\,000 \text{ rubles.}$$

Taking into account that $C_{\Sigma_2} < C_{\Sigma_1}$, we come to the conclusion that it is feasible to even only one development, and the optimal time for putting the new type of computers into production will be t_2 = 11.67 years.

The question of the feasibility of developing a large number of new types was not considered in this example.

5. PROBLEMS OF SELECTING THE OPTIMAL SET (TYPES) OF TECHNICAL DEVICES

5.1. Statement of the Problem

In determining the basic characteristics of any technical device we encounter problems, the essence of which we shall first consider in a very simple example.

The distribution function of the non-stop range of transport airplanes is known from experience (it can be obtained by statistical analysis of data). The question arises of creating one type of airplane which can fly any distance necessary in practice; two types, one of which can fly any distance and another whose flight range is less and, therefore, is simpler and cheaper, both in production and in exploitation; or several types.

The more types of airplanes, the less are expenditures for each flight, as selection of the required type of airplanes depends on the required distance. But at the same time, expenditures increase for development, testing and putting the airplane into production (instead of one type, several types must be developed), the specific cost of each type of airplane increases due to the reduced volume of production of airplanes of each type, and, finally, specific expenditures for the exploitation of each type of airplane increase.

Obviously, there is an optimum number of types (set)of airplanes and an optimal flight range for each of them, at which summary expenditures are minimized.

Such a problem also arises in selecting optimal basic characteristics of machines: lathes, milling machines, and the like, forge and press equipment, etc. The approach to solving such problems is described in works [24, 32, 17], where an attempt is made to work out a method of determining the optimal set of parameters for machinery.

An analogous problem can be encountered in determining a set of load-carrying capacities for trucks and cars, river and sea vessels, basic parameters of computers,

means of communication, and, in general, any mass technical devices.

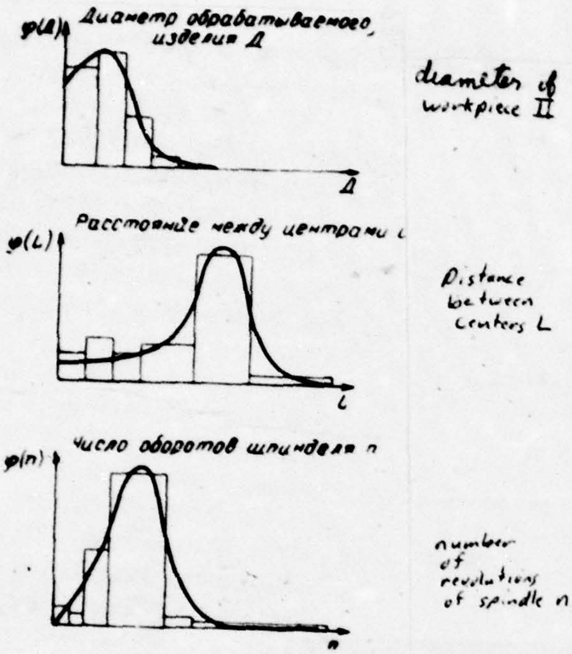

Figure 5.1.1. Kind of $\varphi(x)$ for lathes.

We formulate a mathematical problem.

The function of demand is given from a certain argument: integral $F(x)$ or $\varphi(x)$. Fig. 5.1.1 shows those functions for lathes, taken from [32].

The function of the production cost of one technical device is given from this argument $C_0(x)$, the cost of development, testing and putting into production of a new type of technical device $C_p(x)$ and the cost of exploitation or storage of the device per unit of time $C_x(x)$. Several examples of such functions have been given before.

The optimal number of types of technical devices must be determined and those values of their arguments at which summary expenditures are minimized.

Let N types be selected, the arguments of which are $x_1, x_2, \ldots, x_N$, and each device of k-type is applied in the range of the argument from x_{k-1} to x_k. Then we can write the following expression for summary expenditures, ignoring failure of technical devices in proportion to exploitation:

$$S_N = \sum_{k=0}^{N} [F(x_{k+1}) - (F(x_k)]^{\mu} C_0(x_{k+1}) + \\ + \sum_{k=0}^{N} C_p(x_{k+1}) + \int_0^T \sum_{k=0}^{N} C_x(x_{k+1}) [F(x_{k+1}) - F(x_k)] \, dt. \tag{1}$$

We must determine the set of x_k, including N, which minimizes S_N.

The above described problem is univariate, as only one argument is present here. Other varieties of problems of this type are also possible; a typical classification is shown in Fig. 5.1.2.

In practice we encounter bivariate problems and problems of even higher dimensionality. In the example of transport aviation we can simultaneously select flight range and load-carrying capacity; for lathes — maximum diameter and length of the work piece, number of revolutions of the spindle, longitudinal delivery and drive power; for stamping and press machinery — force of the press, number of double lines per minute, etc.; for automobiles — load capacity, speed, etc. These problems are more complex.

In some cases the types of technical devices can be fixed (i.e. already developed and in production), and the problem will consist of finding the parameters of additional types of devices with a number of fixed parameters.

The example given relates to a case when the argument of devices of a given type must be no less than a required level. A slightly different statement of the problem is also possible: if the argument of a device of a certain type is less than given, certain additional expenditures arise, the value of which can be a function of the difference between arguments. For example, if an airplane has a maximum range (argument) less than given, it can fulfill the task with intermediate landing, which requires additional expenditures. Or insufficient load-carrying capacity of a transport vehicle can require preliminary disassembly, loading, and reassembly after transfer, which also entails additional expenditures. In this case it is necessary to select two sets: a set of arguments of technical devices and a set of arguments until which they are to be used.

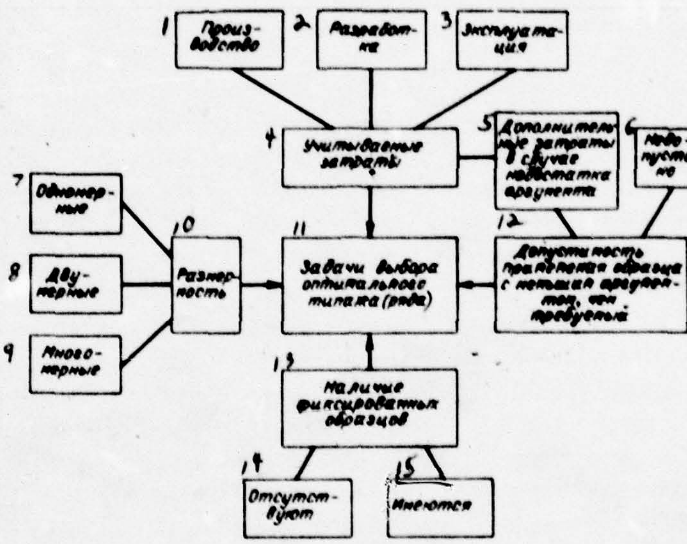

Figure 5.1.2. Classification of problems of selecting optimal sets.

1. Production
2. Development
3. Exploitation
4. Calculated expenditures
5. Additional expenditures in case of insufficiency of argument
6. Inadmissible
7. Univariate
8. Bivariate
9. Multivariate
10. Dimensionality
11. Problems of selecting optimal types (set)
12. Admissibility of using a model with lesser argument than required
13. Presence of fixed models
14. Absent
15. Present

Finally, problems of selecting optimal sets of technical devices can be not only static, but also dynamic. In dynamic problems the process of feasible change of sets in time is considered. They are a combination of problems of selecting optimal sets and generalized problems of equipment replacement.

According to their character, problems of selecting optimal sets are distribution problems; however, they differ by the presence of nonfixed values of the argument, as well as a nonfixed number of arguments, which considerably complicates their solution.

In problems of selecting optimal sets additional conditions connected with standardization can be considered.

In particular, this situation can be imposed: all selected values of x_k must be terms of geometric progression with the denominator $\sqrt[n]{10}$. Then the problem is greatly simplified.

In this case

$$x_k = a(\sqrt[n]{10})^k \tag{2}$$

and

$$n = \frac{N}{\log \frac{x_N}{x_0}}. \tag{3}$$

A univariate problem amounts to selecting one variable N, minimizing functional (1), which, considering the discrete nature of N (1, 2, 3,...), presents no difficulties.

The problem is even more simplified if we apply the condition of selecting values of arguments from existing sets of preferred numbers (for example, R_5, R_{10}, R_{20} and R_{40}). Then the problem is reduced to calculating S_N with $\frac{x_k}{x_0}$, corresponding to various sets of preferred numbers, and selecting the set with which S_N is minimal.

However, the application of additional conditions can considerably increase expenditures S_N. Therefore, in all cases it is necessary to solve the problem

without additional conditions, to compare with S_N in the case of fulfillment of all these conditions, and then make a decision depending on the difference in S_N.

Later we shall describe very simple problems of selecting optimal sets, a general method of solving univariate problems and an approach to their solution in complicated cases.

5.2. Simplest Problems of Selecting Optimal Sets

Let us consider a very simple univariate case, for which we must obtain analytical formulae.

Let

$$C_0 = ax, \tag{1}$$

i.e. the cost of production of the device is proportional to the argument (we note that dependences similar to this are quite frequently encountered in actual practice);

$$F(x) = b(x - x_0) \text{ при } x_0 < x < x_N;$$
$$F(x) = 0 \text{ при } x < x_0; \qquad \text{[при = with]} \tag{2}$$
$$F(x) = b(x_N - x_0) \text{ при } x > x_N,$$

i.e. the differential function of demand in the argument is constant in a certain range of the argument and equal to zero beyond its limits;

$$C_p(x) = cx, \tag{3}$$

i.e. the cost of development of the device is proportional to the value of the argument.

For the present we ignore expenditures for exploitation, i.e. we assume $C_x(x) = 0$. We also assume $\mu = 1$.

Solution of the problem consists of the following: We set the number of types, we select by ordinary methods optimal values of arguments for this case and we cal-

culate expenditures. Repeating this procedure for different numbers of types, we also select the optimal number of types.

If the type of devices is one, then evidently the value of its argument will be x_N, and summary expenditures in this case are calculated according to formula

$$S = b(x_N - x_0)ax_N + cx_N. \tag{4}$$

We write this equation in dimensionless form:

$$\tilde{S} = \frac{S}{abx_N^2} = 1 - \tilde{x}_0 + k, \tag{5}$$

where

$$k = \frac{c}{abx_N}; \tag{6}$$

$$\tilde{x}_0 = \frac{x_0}{x_N}. \tag{7}$$

In the future we shall indicate the value $\tilde{x}_0$ as x_0.

If there are two types of devices, then the argument of the first type of device is determined from the condition of minimization of summary expenditures; for the second type it is equal to x_N.

In this case summary expenditures

$$\tilde{S} = x_1(x_1 - x_0) + (1 - x_1) + kx_1 + k. \tag{8}$$

Here x_0 and x_1 designate the ratios of these values to x_N. Having equalized the first derivative of S to zero and having solved the resulting equation, we obtain

$$x_1 = \frac{1 + x_0 - k}{2}; \tag{9}$$

expenditures for this case are calculated according to formula

$$\tilde{S}_0 = \frac{3 - 2x_0 - x_0^2 + k}{4} + \frac{3k + kx_0 - k^2}{2}. \tag{10}$$

The first term indicates expenditures for production $(\tilde{S}_n)$, , the second — expenditures for development $(\tilde{S}_p)$.

TABLE 5.2.1

Number of types N	Expenditures for production $\tilde{S}_{\text{п}}$	Expenditures for development $\tilde{S}_{\text{р}}$	Optimal values of arguments
1	$1 - x_0$	k	1
2	$\frac{3 - 2x_0 - x_0^2 + k^2}{4}$	$\frac{3 + x_0 - k}{2}k$	$\frac{1 + x_0 - k}{2}$; 1
3	$\frac{6 - 3x_0 - 3x_0^2 + 9k^2}{9}$	$\frac{6 + 3x_0 - 6k}{3}k$	$\frac{1 + 2x_0 - 3k}{3}$; $\frac{2 + x_0 - 3k}{3}$, 1
4	$\frac{10 - 4x_0 - 6x_0^2 + 40k^2}{16}$	$\frac{10 + 6x_0 - 20k}{4}k$	$\frac{1 + 3x_0 - 6k}{4}$, $\frac{2 + 2x_0 - 8k}{4}$, $\frac{3 + x_0 - 6k}{4}$, 1
5	$\frac{15 - 5x_0 - 10x_0^2 +}{25} \rightarrow$ $\rightarrow \frac{+ 125k^2}{25}$	$\frac{15 + 10x_0 - 50k}{5}k$	$\frac{1 + 4x_0 - 10k}{5}$, $\frac{2 + 3x_0 - 15k}{5}$, $\frac{3 + 2x_0 - 15k}{5}$, $\frac{4 + x_0 - 10k}{5}$, 1
6	$\frac{21 - 6x_0 - 15x_0^2 +}{36} \rightarrow$ $\rightarrow \frac{+ 315k^2}{36}$	$\frac{21 + 15x_0 - 105k}{6}k$	$\frac{1 + 5x_0 - 15k}{6}$, $\frac{2 + 4x_0 - 24k}{6}$, $\frac{3 + 3x_0 - 27k}{6}$, $\frac{4 + 2x_0 - 24k}{6}$, $\frac{5 + x_0 - 15k}{6}$, 1

The problem is also solved analogously in those cases when the number of types of devices is three or more.

Without dwelling on uncomplicated operations, we present resultant formulae in Table 5.2.1.

We recall that $\tilde{S} = \tilde{S}_n + \tilde{S}_p$, and values of arguments are reduced to x_N.

It is interesting to trace the change in relative expenditures for production in a very simple case ($x_0 = 0$; $k = 0$). From the table it can be seen that expenditures are reduced, as is shown in Fig. 5.2.1. The limit of expenditures at $N \longrightarrow \infty$, as it is not hard to see, will be 0.5.

Relative expenditures for development $\tilde{S}_p/k$ at $x_0 = 0$ and low k increase linearly with rise of N. With significant values of k this dependence can differ significantly from linear.

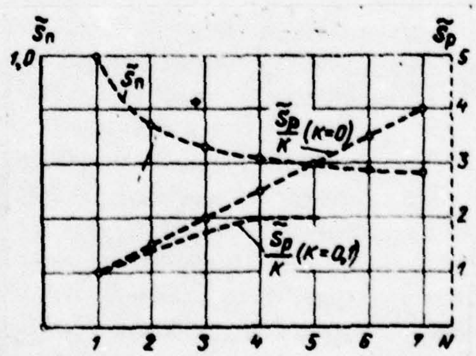

Figure 5.2.1. Change in expenditures with change in the number of types of technical devices.

Example 5.2.1. Analysis of statistical data shows that the required non-stop flight range of airplanes is equally distributed from $x_0 = 100$ to $x_N = 1000$ km. We must select the optimal set of

airplane ranges, if the airplane total is 1,800; cost of production of an airplane with a range of 1,000 km C_0 is 100,000 rubles, and in other ranges it is proportional to the range; the cost of development, testing and putting into production an airplane with a range of 1,000 km C_p is 10,000,000 rubles and in other ranges is proportional to the range.

In this case, based on (1) and (2), we obtain

$$a = \frac{100,000}{1000} = 100 \text{ rub/km};$$

$$b = \frac{1800}{1000 - 100} = 2 \text{ airplanes/km};$$

$$c = \frac{10,000,000}{1000} = 10,000 \text{ rub/km}.$$

According to formulae (6) and (7)

$$k = \frac{10\,000}{100 \times 2 \times 1000} = 0{,}05;$$

$$x_0 = \frac{100}{1000} = 0{,}1.$$

Using formulae from Table 5.2.1, we fill in Table 5.2.2.

TABLE 5.2.2

N	$\tilde{S}_п$	$\tilde{S}_р$	$\tilde{S}=\tilde{S}_п+\tilde{S}_р$	Optimal presented values of arguments
1	0,900	0,050	0,950	1,000
2	0,697	0,076	0,773	0,525 1,000
3	0,633	0,100	0,733	0,350 0,650 1,000
4	0,602	0,120	0,722	0,250 0,450 0,700 1,000
5	0,589	0,135	0,724	0,180 0,310 0,490 0,720 1,000

From the table it can be seen that the optimal set is a set of four airplanes with flight ranges of 250, 450, 700 and 1000 km. Let us note that the extremum is very mildly sloping and without any great loss a set of three airplanes with ranges of 350, 650 and 1000 km can be taken if certain additional considerations in favor of reducing the number of types of airplanes are present.

As seen from the example, selection of the optimal set can considerably (up to 20% in these conditions) reduce expenditures.

Above, expenditures for exploitation were not calculated. In a number of cases they can be represented in the form of the following dependence:

$$C_z(x_k) = Dx_k[F(x_k) - F(x_{k-1})]. \tag{11}$$

Then

$$\int_0^T C_z(x_k)\,dt = bDTx_k(x_k - x_{k-1}). \tag{12}$$

From which

$$S_{np} = \frac{S_N}{(ab + bDT)\,x_N^2} = \sum_{k=0}^{N}\left(\frac{x_{k+1}}{x_N} - \frac{x_k}{x_N}\right)\frac{x_{k+1}}{x_N} + k^*\sum_{k=1}^{N}\frac{x_k}{x_N}, \tag{13}$$

where

$$k^* = \frac{c}{b(a + DT)\,x_N}. \tag{14}$$

Thus, the problem is reduced to the one already solved, but instead of k, we must substitute k^*, determined according to formula (14).

Let us now consider a case when there are fixed types of devices. The method of solving the problem does not differ essentially from the one described. As in the preceeding case, equations are written for cost from x_0 to $x_{\Phi 1}$; from $x_{\Phi 1}$ to $x_{\Phi 2}$, etc., where $x_{\Phi 1}$, $x_{\Phi 2}$, ... are fixed values of arguments; calculations are made of optimal values of intermediate values of arguments with 1, 2, etc., inter-

mediate points of cost for these cases and summary costs with different combinations of the number of intermediate points.

Let us consider the following simplified problem. Let

$$\left.\begin{array}{ll} C_0(x)=ax; & \\ F(x)=b\,[2x_N x - x^2] & \text{при } 0 \leqslant x \leqslant x_N; \\ F(x)=0 & \text{при } x<0; \\ F(x)=bx_N^2 & \text{при } x \geqslant x_N; \\ C_p(x)=g_1; & \\ C_\lambda(x)=0; & \\ \mu=1. & \end{array}\right\} \tag{15}$$

[при = with]

We must select the optimal set.

This problem is solved by the same method as in the preceeding case. We fix N, we calculate the optimal set with fixed N and calculate S_{N_0}. Sequentially increasing N, we find its value at which S_{N_0} is minimum.

With $N = 1$ we obtain

$$S_N=abx_N^2+g_1. \tag{16}$$

With $N = 2$

$$S_N=ab\,[3x_N x_1^2-x_1^3+x_N^3-2x_N^2 x_1]+2g_1.$$

Differentiating by x_1 and equalizing the derivative to zero, we obtain

$$6x_N x_1-3x_1^2-2x_N^2=0.$$

from which

$$x_1=x_N\left(1-\frac{1}{\sqrt{3}}\right)\approx 0{,}4226x_N. \tag{17}$$

The second root, as it has no physical meaning, we discard.

The problem is solved analogously with $N = 3$, 4, etc. Results of solution are given in Table 5.2.3.

TABLE 5.2.3

Number of types N	Summary expenditures s_N	Optimal values of arguments reduced to x_N
1	$abx_N^3 + g_1$	1
2	$0{,}6151 \cdot abx_N^3 + 2g_1$	0,4226; 1
3	$0{,}5092 \cdot abx_N^3 + 3g_1$	0,2639; 0,5750; 1
4	$0{,}4606 \cdot abx_N^3 + 4g_1$	0,1909; 0,4044; 0,6561; 1
5	$0{,}4329 \cdot abx_N^3 + 5g_1$	0,1493; 0,3118; 0,4934; 0,7075; 1
6	$0{,}4199 \cdot abx_N^3 + 6g_1$	0,1225; 0,2535; 0,3961; 0,5554; 0,7433; 1

Using data from this table, it is not difficult to calculate the selection of optimal N and x_k.

Example 5.2.2. Analysis of statistical data shows that the nonstop flight range of airplanes is distributed from 0 to x_N = 1,000 km according to linear differential law with maximum at minimum range and frequency equal to zero at maximum.

The total number of airplanes must be 1800.

Cost of production of an airplane C_0 with a range of 1,000 km is 100,000 rubles, and for other ranges it is proportional to the range; cost of development of an airplane C_P is independent of the flight range, 5,000,000 rubles.

We must select the optimal set of ranges.

Based on formulae (15) and (1) we obtain

$$b = \frac{1800}{1000^2} = 0.0018;$$

$$a = \frac{100{,}000}{1000} = 100 \text{ rub/km}.$$

Using formulae from Table 5.2.3, we fill in Table 5.2.4.

From the table it can be seen that the set consisting of four types of airplanes with ranges of about 200, 400, 650 and 1000 km will be optimal.

TABLE 5.2.4

Number of types N	Summary expenditures S_N	Optimal values of arguments reduced to x_N
1	$100\times 0{,}0018\times 1000^2+5000000=$ $=185000000$	1
2	$0{,}6151\times 100\times 0{,}0018\times 1000^2+$ $+2\times 5000000=120\,718\,000$	0,4226; 1
3	$0{,}5092\times 100\times 0{,}0018\times 1000^2+$ $+3\times 5000000=106\,656\,000$	0,2639; 0,5750; 1
4	$0{,}4606\times 100\times 0{,}0018\times 1000^2+$ $+4\times 5000000=102\,908\,000$	0,1909; 0,4044; 0,6551; 1;
5	$0{,}4329\times 100\times 0{,}0018\times 1000^2+$ $+5\times 5000000=102\,922\,000$	0,1493; 0,3118; 0,4993; 0,7075; 1

5.3. GENERAL METHOD OF SOLVING UNIVARIATE PROBLEMS

Let us consider first of all a problem of minimizing the sum

$$S_N=\sum_{k=0}^{N}[F(x_{k+1})-F(x_k)]\,C_0(x_{k+1}), \qquad (1)$$

having generalized it to a case with some types of devices already in existence, i.e. m points already fixed.

Then we consider
$$S_N=\sum_{k=0}^{N=m+n}[F(x_{k+1})-F(x_k)]\,C_0(x_{k+1}),$$
$$\alpha=x_0<x_1<\ldots<x_{N+1}=\beta. \qquad (2)$$

If there are no fixed points, then the problem can be solved by dynamic programming using sequential calculation

$$S_j(x) = \min\{S_{j-1}(x) + [F(x) - F(y)]\, C_0(x)\},$$
$$a < y < x, \quad j = 1, 2, \ldots, N, \tag{3}$$

where

$$S_0(x) = [F(x) - F(a)]\, C_0(x), \quad x \in (a, \beta).$$

Completion of one step, i.e. calculating $S_j(x)$ in all nodes of ε-network (number of nodes M), takes c_1M operations (c_1 — some constant) and M cells for storing intermediate results. Solution of the entire problem requires c_1NM^2 operations and NM cells.

In the presence of m fixed points in each of $m + 1$ fixed segments, n steps are completed according to formula (3). After this, all possible variants of arrangement of n points in $m + 1$ segments are sorted and each time S_N is calculated. Then the arrangement where S_N reaches minimum will be the sought arrangement.

This takes $c_1 n \sum_{l=1}^{m+1} M_l^2 + c_2 c_{n+m}^n$ operations and nN memory cells.

Here M_l is the number of nodes of ε-network in $[x_{i_{l-1}}; x_{i_l}]$; $\sum_{l=1} M_l = M$; c_2 — some constant.

This method of solution is extremely time consuming. V. T. Dement'yev [20] has suggested a less time consuming method for solving this problem, valid if $F(x)$ and $C_0(x)$ are positive nondecreasing functions.

Let us note that $F(x)$ is always a nondecreasing functions; as for $C_0(x)$, in practically all cases it will also be a nondecreasing function. The cost of the technical device does not decrease with increase of the argument (distance, load-carrying capacity, etc.).

In the above indicated assumptions regarding $F(x)$ and $C_0(x)$, function (2) is an upper Riemann-Stieltjes integral sum. The following theorem, proved by V. T.

Dement'yev, is valid for this sum.

With any nondecreasing, positive functions $F(x)$ and $C_0(x)$, determined in segment $[\alpha, \beta]$, the sequence produced by minimums of upper Riemann-Stieltjes integral sums,

$$S_j = \min_{\{x_k\}} \sum_{k=0}^{j} [F(x_{k+1}) - F(x_k)] C_0(x_{k+1}),$$
$$\alpha = x_0 < x_1 \leq \ldots < x_{j+1} = \beta, \quad j = 0, 1, 2, \ldots,$$

is a monotonically decreasing convex:

$$S_j > S_{j+1} \text{ and } S_j - 2S_{j+1} + S_{j+2} \geq 0. \tag{4}$$

From this theorem it follows that in each fixed segment $[x_{i_{l-1}}, x_{i_l}]$ is fulfilled inequality (4)

$$\Delta_{j,l} \geq \Delta_{j+1,l} \text{ with } j = 1, 2, 3, \ldots, \tag{5}$$

where

$$\Delta_{j,l} = S_{j,l} - S_{j-1,l};$$
$$S_{j,l} = \min S_j^* \text{ in segment } [x_{i_{l-1}}, x_{i_l}].$$

Inequality (5) indicates that reduction of the minimum of function S from the addition of the next point is not greater than reduction of S from addition of the preceeding point. Using this fact, we can obtain the following solution algorithm for this problem.

In the first step we calculate $\Delta_{1,l} (l = 1, 2, \ldots, m+1)$, i.e. one step is completed in each segment $[x_{i_{l-1}}, x_{i_l}]$ according to formula (3) with $j = 1$.

Then a maximum is sought among $\Delta_{1,l}$. If this is $\Delta_{1,p}$, then the first point is placed in p-interval.

With distribution of the second point, it is necessary to find the segment in which the greatest reduction is achieved from addition of this point. From inequality (5) it follows that this segment can be either the segment in which

$\Delta_{1,l}\ (l \neq p)$ — next in value after $\Delta_{1,p}$ (we designate this by $\Delta_{1,q}$), or p-segment, if $\Delta_{2,p} > \Delta_{1,q}$.

In order to explain this, we must make one more step according to formula (3) in p-segment and compare the obtained $\Delta_{2,p}$ with $\Delta_{1,q}$. If $\Delta_{1,q}$ is greater, then the second point is placed in q-interval, if $\Delta_{2,p}$ is greater, then both points are set in p-interval.

We must proceed analogously in subsequent steps, i.e. complete the next step according to formula (3) in that interval in which the preceeding point was set, and then compare the obtained Δ with the maximum from the rest.

This process is repeated untill all n points are distributed.

Realization of the described algorithm only requires

$$c_1 \sum_{l=1}^{m+1} M_l^2 r_l$$

operations and $\sum_{l=1}^{m+1} M_l r_l$ memory cells, where r_l — number of steps completed in intervals $(x_{i_{l-1}}, x_{i_l}]$; $1 \geq r_l \geq n$; $\sum_{l=1}^{m+1} r_l = N$.

Using the described algorithm, the problem is easily solved with a computer, and then with fixed N it is easy to obtain the optimal set, i.e. the set of values $x_1, x_2, \dots, x_N$, minimizing expenditures for the production of models S_N.

It remains only to select the optimal value of N. This problem is solved as follows.

If $C_p(x)$ — increasing function of x, then, considering

$$S^*_N = S_N + \sum_{k=1}^{N} C_p(x), \tag{6}$$

with fixed N we can use the algorithm described above and select the optimal set. This requires calculating S^*_N. With a sequential increase of N, it is necessary to stop at that value at which minimum S^*_N is ensured.

5.4. Selection of an Optimal Set in Multivariate Cases

Let us consider first the problem of selecting a bivariate set (for example, distance x and load-carrying capacity y).

In this case, the function of demands will be a function of two arguments $F(x,y)$.

It is more convenient to put it in differential form $\varphi(x, y)$; then

$$F(x, y)=\iint \varphi(x, y)\, dx\, dy.$$

This is a nondecreasing function of x and y.

The type of technical device is defined by two arguments x_k and y_k. In this case, the cost of production of the device is $C_0(x,y)$. Cost of development, testing and putting into production of a new type of device $C_p(x, y)$(nondecreasing function of x and y) and the cost of exploitation of the technical device per unit of time $C_x^{\cdot}(x, y)$ (nondecreasing function of x and y) are usually also functions of two arguments.

Summary expenditures are determined by the following dependence:

$$\begin{aligned} S_N = & \sum_{k=0}^{N} [F(x_{k+1};\, y_{k+1}) - F(x_k;\, y_k)]\, C_0(x_{k+1};\, y_{k+1}) + \\ & + \sum_{k=1}^{N} C_p(x_k;\, y_k) + \int_0^T \sum_{k=0}^{N} [F(x_{k+1};\, y_{k+1}) - \\ & - F(x_k;\, y_k)]\, C_x(x_{k+1};\, y_{k+1})\, dt. \end{aligned} \qquad (1)$$

We must determine the set $x_k;\, y_k$, minimizing S_N.

As in this problem the technical device can be used only when arguments do not exceed their permissible limits for the given device, it is necessary to fulfill one more condition.

If $\varphi(x, y)$, is given, then with $\varphi(x, y)=0$ we obtain $y=\varphi_2(x)$. It is necessary to fulfill the following condition with all k:

$$\begin{aligned} & \max \varphi_2(x) < \max y_j; \\ & x_0 < x < x_k;\ k < j < N. \end{aligned} \qquad (2)$$

One effective method of solving this problem at present is random search. We set $N = 1, 2, 3$, etc. With each N we obtain $N - 1$ random numbers, evenly distributed in the interval from x_0 to x_N, and, having added to it x_N, after arrangement in increasing order, we determine the variant of the set according to argument x.

Then, if function $y=\varphi_2(x)$ is nondecreasing, y_k is determined as a random number from the group of numbers evenly distributed in the interval $0 \div y=\varphi_2(x_k)$ and $y_N=\varphi_2(x_N)$.

If function $y=\varphi_2(x)$ is decreasing, then as y_k is taken $\varphi_2(x_{k-1})$.

Usually (function $\varphi_2(x)$ has maximums or minimums) the simplest algorithm of determining y_k will be selecting it from the group of evenly distributed numbers in interval $0 \div \max\limits_{[x_0;\ x_k]} \varphi_2(x)$ with subsequent verification of condition (2) and increase, when necessary, of y_k until fulfillment of this condition.

Calculations are conducted with fixed values of N, storing the best result with least value of S_N.

Conducting a series of calculations with different N, we select the optimal value of N and its corresponding optimal set.

As the simplest bivariate problem, let us consider the following.

Let

$$\begin{aligned} &F(x,\ y)=bxy \text{ при } && 0<x<x_N, \\ & && 0<y<y_N; \\ &F(x,\ y)=bx_Ny_N \text{ при } && x\geqslant x_N, \\ & && y\geqslant y_N; \\ &F(x,\ y)=0 \text{ при } && x<0, \\ & && y<0, \end{aligned} \tag{3}$$

[при = with]

i.e. density of the function of demand in the area limited by the axes of coordinates and straight lines $x=x_N$ and $y=y_N$, is constant.

Let

$$C_0(x,\ y)=axy; \tag{4}$$

$$C_p(x,\ y)=C_p; \tag{5}$$

$$C_x(x)=0. \tag{6}$$

We must select the optimal set of technical devices.

If there is one type of technical device, then its arguments must be x_N and y_N, and then

$$S_N = F(x, y) C_0 (x_N; y_N) + C_p = abx_N^2 y_N^2 + C_p. \tag{7}$$

If there are two types of technical devices, then

$$S_N = ab(x_1^2 y_1^2 - x_1 y_1 x_N y_N + x_N^2 y_N^2) + 2C_p. \tag{8}$$

Differentiating this equation by x_1 and y_1, and equalizing derivatives to zero, we obtain two identical equations

$$2x_1 y_1^2 - y_1 x_N y_N = 0, \tag{9}$$

from which it follows that

$$x_1 = \frac{x_N y_N}{2y_1}. \tag{10}$$

Evidently $y_1 < y_N$, therefore

$$x_1 \quad \frac{x_N}{2}.$$

On the other hand $x_1 < x_N$, therefore

$$\frac{x_N}{2} < x_1 < x_N \tag{11}$$

and by analogy

$$\frac{y_N}{2} < y_1 < y_N. \tag{12}$$

Therefore, x_1 and y_1 can have any values within the limits indicated above, with the observance of ratio (10) between them.

In this case

$$S_N = \frac{3}{4} abx_N^2 y_N^2 + 2C_p. \tag{13}$$

If additional condition $\frac{x}{x_N}=\frac{y_1}{y_N}$, is applied, then we obtain

$$x_1=\frac{x_N}{\sqrt{2}};\quad y_1=\frac{y_N}{\sqrt{2}}. \tag{14}$$

Analogously, we also obtain corresponding dependences for large N.

In general form, these formulae are written as follows:

$$\tilde{S}_N=\frac{S_N^{\cdot}}{abx_N^2y_N^2}=\frac{1+N}{2N}+KN, \tag{15}$$

where

$$K=\frac{C_p}{abx_N^2y_N^2}; \tag{16}$$

$$x_k=\frac{kx_Ny_N}{Ny_k};\quad \frac{k}{N}x_N<x_k<x_{k+1}. \tag{17}$$

The values of arguments with additional conditions $\frac{x_k}{x_N}=\frac{y_k}{y_N}$ will be optimal with

$$x_k=\frac{x_N\sqrt{k}}{\sqrt{N}};\quad y_k=\frac{y_N\sqrt{k}}{\sqrt{N}}. \tag{18}$$

Conducting calculations with different N, we select one value at which S is minimal.

<u>Example 5.4.1</u>. Let demands for air transportation be evenly distributed by distance from 0 to x_N = 1000 km and by load-carrying capacity from 0 to $y_N=10$ т ; 2,000 airplanes are required. The production cost of an airplane is the following function of its flight range (km) and load-carrying capacity (tons):

$$C_0(x,\ y)=10xy.$$

The cost of development of the airplane C_p= 10,000,000 rubles. Select the optimal set of ranges and load-carrying capacities.

In this case, on the basis of (3) and (4)

$$b = \frac{2000}{1000 \times 10} = 0.2;$$

$$a = 10.$$

Using formula (16), we obtain

$$K = \frac{10 \cdot 10^6}{10 \cdot 0{,}2 \cdot 1000^2 \cdot 10^2} = 0{,}05.$$

Using formula (15), we till in Table 5.4.1.

Thus, it is necessary to have only three types of airplanes.

TABLE 5.4.1.

Number of types N	Summary expenditures $З_N$
1	$\frac{1+1}{2 \times 1} + 1 \times 0{,}05 = 1{,}05$
2	$\frac{1+2}{2 \times 2} + 2 \times 0{,}05 = 0{,}85$
3	$\frac{1+3}{2 \times 3} + 3 \times 0{,}05 = 0{,}817$
4	$\frac{1+4}{2 \times 4} + 4 \times 0{,}05 = 0{,}825$

We apply the additional condition of proportionality of range and load-bearing capacity. Then according to (18), we obtain

$$x_1 = \frac{1000\sqrt{1}}{\sqrt{3}} \cong 580 \text{ км}; \quad y_1 = \frac{10\sqrt{1}}{\sqrt{3}} \cong 5{,}8 \text{ т};$$

$$x_2 = \frac{1000\sqrt{2}}{\sqrt{3}} \cong 815 \text{ км}; \quad y_2 = \frac{10\sqrt{2}}{\sqrt{3}} \cong 8{,}1 \text{ т};$$

$$x_3 = 1000 \text{ км}; \quad y_3 = 10 \text{ т}.$$

From the data in Table 5.4.1, it is evident that selection of the optimal set of load-carrying capacities and ranges of airplanes can considerably reduce expenditures (in this case more than 20%).

5.5. Selection of an Optimal Set in the Presence of Additional Expenditures with an Insufficient Argument

Above we considered cases when a technical device of a given type can be used only with arguments which are less than those characterizing this type and greater than the preceeding type.

In practice, we can encounter cases when fulfillment of a given condition is not mandatory. For example, an airplane with maximum non-stop flight range of 1,000 km can also fly for distances over 1,000 km, but making landings, which is connected with additional expenditures.

We shall, as before, call x_k the value of the argument characterizing this type. The value of the argument until which the given type is used we sall call z_k.

Having designated the function of demands $F(x)$, cost of development, testing and putting into production of a given type of device $C_p(x_k)$, , and cost of production of one device $C_0(x_k)$, , we shall consider that the cost of exploitation of one device per unit of time is $C_x(x_k)$, and additional expenditures for one device per unit of time $C_\pi\left(\frac{z}{x_k}\right)$.

Taking into account the above, we write formula (5.1.1) as follows (for simplicity we assume $\mu = 1$):

$$S_N=\sum_{k=0}^{N}[F(z_{k+1})-F(z_k)]\,C_0(x_{k+1})+$$
$$+\sum_{k=0}^{N}C_p(x_{k+1})+\int_0^T\sum_{k=0}^{N}[F(z_{k+1})-F(z_k)]\,C_x(x_{k+1})\,dt+$$
$$+\int_0^T\sum_{k=0}^{N}\int_{x_k}^{z_k}\varphi(x)\,C_\pi\left(\frac{z}{x_k}\right)dx\,dt. \qquad (1)$$

The last term here also represents additional expenditures.

The problem consists of determining the set x_k, z_k, at which S_N reaches a minimum.

Then

$$\varphi(x)=b \text{ at } 0<x<x_N;$$
$$\varphi(x)=0 \text{ at } x<0 \text{ and at } x>x_N. \tag{11}$$

We must select x_k for a technical device, assuming that there will be only one type.

Evidently in this case $x_k=x_N$.

Expression (5) takes the following form:

$$S_N=bx_N\left(ax_k+\int_0^T ex_k dt\right)+cx_k+$$
$$+\int_{x_k}^{x_N} b\left(\frac{x}{x_k}-1\right)dx\int_0^T k_n dt=x_k\left[bx_N(a+eT)+c-\right.$$
$$\left.-\frac{3}{2}bk_nT\right]+\frac{bk_nTx_N^2}{2x_k}-bk_nTx_N. \tag{12}$$

Differentiating by x_k, equalizing the derivative to zero and solving the resulting equation, we find the following formula for calculating the optimal value of x_k:

$$x_k=x_N\sqrt{\frac{k_nT}{2x_N(a+eT)+2\frac{c}{b}+2k_nT}}. \tag{13}$$

If as a result of the calculations we find that $x_k>x_N$, then we must assume $x_k=x_N$.

<u>Example 5.5.2.</u> Determine the optimal maximum flight range of an airplane if required distances vary from 0 to 1,000 km. It is assumed there is one type of airplane.

a = 100 rubles/km; c = 10,000 rubles/km; b = 2 planes/km;
eT = 50 rubles/km; k_nT = 30,000 rubles.

According to formula (13) we obtain

$$x_k=1000\sqrt{\frac{30000}{2\cdot 1000(100+50)+2\frac{10000}{2}+2\cdot 30000}}\approx 280 \text{ km}.$$

6. DETERMINATION OF OPTIMAL RELIABILITY (STRENGTH) OF TECHNICAL DEVICES AND RATIONAL MEANS OF PROVIDING IT

6.1. Determination of Optimal Reliability

The greater the reliability of a technical device, the less the expenditures for its exploitation and expenditures connected with eliminating unexpected failures, the fewer demands for spare technical devices. From this point of view, an increase of reliability is unconditionally advisable.

However, as a rule, increased reliability is connected with additional expenditures, which can be extremely high.

From this it is evident that there is some optimal reliability which can be determined on the basis of the minimum of expenditures needed to perform the tasks set for the technical device. It depends on the operating conditions of the technical device. Completely different demands can be made of the reliability of a spacecraft and a stationary computer, not operating a full working day.

On the other hand, optimal reliability depends on material expenditures necessary to increase it. The less these expenditures, the higher it must be. Finally, optimal reliability also depends on the constraints applied to the technical device (for example, on weight, dimensions, etc.).

Let us consider first the connection between the reliability of a technical device and its cost.

The reliability of a technical device depends primarily on the reliability of its components, P_i. The reliability of components can be increased by improving their design, which is connected with an increase of their cost, by preliminary conditioning of components (see § 6.2), which also requires additional expenditures, and, finally, by improving the operating conditions (for example, by lowering temperatures or decreasing mechanical actions). The latter can also be achieved by

It can be solved by the method of random search in the following sequence:

1) we set N;

2) we obtain $N-1$ random numbers from the group evenly distributed from 0 to x_N. Adding x_N to them and arranging them in increasing order, we obtain preliminary z_k;

3) obtaining N random numbers from the same group as the preceeding, we have preliminary set x_k;

4) switching the places of x_k and z_k from the preliminary set so that $z_k \geqslant x_k$, we obtain final sets z_k and x_k for this attempt and we calculate S_N;

5) we repeat the cycle from 2) to 4), storing the best variant according to S_N

6) we compare results with those relating to the case of $z_k = x_k$.

Consideration of practical examples shows that $C_{\text{д}}\left(\frac{z}{x_k}\right)$ can have two different characters.

In the first case this is a stepped function

$$\begin{aligned} C_{\text{д}}\left(\frac{z}{x_k}\right) &= 0 \text{ при } \frac{z}{x_k} < 1; \\ C_{\text{д}}\left(\frac{z}{x_k}\right) &= C^*_{\text{д}} \text{ при } 1 < \frac{z}{x_k} < 2; \\ C_{\text{д}}\left(\frac{z}{x_k}\right) &= nC^*_{\text{д}} \text{ при } n < \frac{z}{x_k} < n+1. \end{aligned} \tag{2}$$

[при = with]

In the second case, it is a function of the following kind:

$$\begin{aligned} C_{\text{д}}\left(\frac{z}{x_k}\right) &= 0 \text{ при } \frac{z}{x_k} < 1; \\ C_{\text{д}}\left(\frac{z}{x_k}\right) &= k_{\text{д}}\left(\frac{z}{x_k} - 1\right) \text{ при } \frac{z}{x_k} \geqslant 1. \end{aligned} \tag{3}$$

[при = with]

It is not hard to see that the second is a boundary case in relation to the first with $n \to \infty$.

In the first case we can use the following heuristic method of solving the problem. From intuitive reasoning it is clear that $\frac{z_k}{x_k}$ will be a whole number. Then formula (1) is rewritten as follows:

$$S_N = \sum_{k=0}^{N} [F(n_{k+1}x_{k+1}) - F(n_k x_k)] [C_0(x_{k+1}) + \int_0^T C_x(x_{k+1}) dt] + \sum_{k=0}^{N} C_p(x_{k+1}) + \sum_{k=0}^{N} [F(n_k x_k) - F(x_k)] \int_0^T C^*_{\text{д}} dt. \quad (4)$$

Now with fixed n the problem is reduced to a univariate problem of selecting the optimal set, which is solved by the methods described above. Setting $n = 1, 2, \ldots$, and by combinations of these values, we stop at the combination where S_N is minimum.

In the second case, we rewrite equation (1) as follows:

$$S_N = \sum_{k=0}^{N} [F(z_{k+1}) - F(z_k)] \left[C_0(x_{k+1}) + \int_0^T C_x(x_{k+1}) dt \right] + \sum_{k=0}^{N} C_p(x_{k+1}) + \sum_{k=0}^{N} \int_{x_k}^{\nu x_k} \varphi(x) \left(\frac{x}{x_k} - 1 \right) dx \int_0^T k_{\text{д}} dt, \quad (5)$$

where

$$\nu = \frac{z_k}{x_k}. \quad (6)$$

If we assume that v is not a discrete value, the problem becomes quite complicated. One possible method of solving it is sequential search according to v, solving the problem of selecting an optimal set for each v.

Let us consider an extremely simple case of solving such a problem. Let

$$F(x) = bx \text{ при } 0 < x < x_N; \quad (7)$$
$$F(x) = 0 \text{ при } x \leq 0;$$
$$F(x) = bx_N \text{ при } x \geq x_N; \quad (8)$$
$$C_0(x) = ax;$$
$$C_p(x) = cx; \quad (9)$$
$$C_x(x) = ex. \quad (10)$$

[при = with]

material expenditures, increasing the weight and dimensions of devices.

To evaluate the reliability of a component, we can usually write

$$P_i = f_i(C_{si}, t_{\tau i}, \tau_i, n_i), \tag{1}$$

where C_{si} — cost of i-component; $t_{\tau i}$ — time of conditioning; τ_i — temperature at which the component operates; n_i — mechanical loads experienced by the component

The reliability of a technical device ca. be increased by redundancy, i.e. the introduction into the design of several k_i instead of one i-component and the introduction of spare parts (ZIP) in the amount of k_{3i} (assuming that replacement will take place quickly). It can also be increased by preventive operations, for example, replacement of components after they have been in operation for a certain period of time t_{3i}.

In addition, the reliability of a technical device can be improved by careful checks of the technical device during its preparation for serial production and during serial production; this is determined to a significant degree by the number of tested articles $n_{\text{и}}$.

Without dwelling on other ways of increasing reliability (for example, changing the basic design of the technical device), we shall list several dependences to which we shall turn later.

The reliability of a technical device can be expressed by the following functional:

$$P = P\{\vec{f}(C_{si}, t_{\tau i}, \tau_i, n_i)\vec{K}, \vec{K}_3, \vec{t}_3, n_{\text{и}}\}, \tag{2}$$

where $\vec{f}; \vec{K}; \vec{K}_3; \vec{t}_3$ — vectors with components f_i, k_i, etc.

The cost of a technical device, if it is movable (transported on some undercarriage), is

$$\begin{aligned} C_1 = \Big\{ & a_1 + a_2 \Big[\sum_i (\nu_1 k_i + \nu_2 k_{3i}) g_i + \mu \sum_i (\nu_3 k_i + \nu_4 k_{3i}) + \\ & + g_{\text{ох}}(\bar{\tau}) + g_{\text{a}}(\bar{n}) + \mu (w_{\text{ох}}(\bar{\tau}) + w_{\text{a}}(\bar{n}) \Big] + \\ & + \nu_5 \sum_i [(k_i + k_{3i}) C_{si} + C_{\tau i} t_{\tau i} (k_i + k_{3i})] + \\ & + C_{\text{ох}}(\bar{\tau}) + C_{\text{a}}(\bar{n}) \Big\} \frac{N_{\text{п}}}{N_{\text{п}} - n_{\text{и}}} \psi(N_{\text{п}}). \end{aligned} \tag{3}$$

Here the first two terms are the cost of the undercarriage, expressed as a function of load capacity (α_1 and α_2 - coefficients of this dependence); the third term is the cost of the device itself; the fourth and fifth - the cost of cooling devices, expressed as a function of the temperature field (vector τ with components τ_i), and shock absorbers, expressed as a function of mechanical loads (vector with components n_i).

Coefficients ν show the increase of weight g, volume w and cost due to additional devices (for example, wiring and panels in electronic devices), μ - coefficient of increase of weight due to the volume of the devices, $N_и$ — total number of technical devices produced, $\Phi(N)$ — function indicating the dependence of the cost of production on its volume.

In more compact form, this formula can be written as follows:

$$C_и = C(\bar{a}, \bar{K}, \bar{K}_з, \bar{C}_в, C_{ох}, C_а, \bar{l}_т, \bar{l}_з, n_и, N)... \tag{4}$$

An analogous dependence also occurs for stationary technical devices, only in this case, coefficients $\bar{\alpha}$ have a different physical meaning (they are connected with the cost of delivery, assembly, and the building in which the device is installed).

Depending on the functional characteristics of the device, constraints can be applied on its weight or dimensions, i.e.

$$Q_{доп} \geqslant \sum_i [(\nu_1 k_i + \nu_2 k_{зi}) g_i + \mu(\nu_3 k_i + \nu_4 k_{зi}) w_i] + g_{ох}(\bar{\tau}) + g_а(n) + \mu[w_{ох}(\bar{\tau}) + w_а(\bar{n})]; \tag{5}$$

$$W_{доп} \geqslant \sum_i [(\nu_3 k_i + \nu_4 k_{зi}) w_i + w_{ох}(\bar{\tau}) + w_а(\bar{n})] \tag{6}$$

or in more compact form

$$Q_{доп} \geqslant Q(\bar{K}, \bar{K}_з, \bar{C}_в, C_{ох}, C_а); \tag{7}$$

$$W_{доп} \geqslant W(\bar{K}, \bar{K}_з, \bar{C}_в, C_{ох}, C_а). \tag{8}$$

Let us consider the question of necessary summary expenditures connected with the volume of reserves and the volume of consumption when technical devices are used to perform a task when the probability of no failure occurring during performance

of the task is P < 1.

These expenditures depend on conditions of the function of technical devices (external model).

We study first of all technical devices used one time, for example, a radiosonde, used to determine the parameters of the atmosphere. In the case of a failure, the radiosonde launch does not give the necessary information and the next one must be launched.

The number of technical devices necessary for fulfilling the task is calculated as follows. Let the probability of fulfilling the task by one device be P. If there are no faulures and one device fulfills the task, by P we must understand P_H — the probability of no-fail operation.

If N devices are selected for performing one task, then the probability of fulfilling task P_N is determined as follows:

$$P_N = 1 - (1 - P)^N. \tag{9}$$

From that, setting the probability of fulfilling one specific task as P_N, we can calculate the number of devices (reserve) necessary for fulfilling the task

$$N = \frac{\log(1 - P_N)}{\log(1 - P)}. \tag{10}$$

This is the number of devices needed to have in reserve, but it differs from their actual consumption. Actually, the task can be fulfilled by the first device, and then the second is not needed, while according to formula (10) it is always necessary to use up N devices.

Let us consider now a case when many similar tasks are performed by identical devices and after fulfillment of each specific task, the consumption of devices for it stops.

Each use of a device accomplishes an average of P tasks. If M devices are used, then on the average PM tasks will be accomplished.

On the other hand, according to formula (9), N operating devices will perform an average of P_N tasks.

Then

$$MP = 1 - (1 - P)^N,$$

from which average consumption is

$$M = \frac{1 - (1 - P)^N}{P}. \tag{11}$$

If the reserve devices allotted for accomplishing the task is unlimited, then $N \to \infty$ and

$$M_\infty = \frac{1}{P}, \tag{12}$$

i.e. with an unlimited reserve, actual consumption in fulfilling the task is a value which is inverse to the probability of fulfilling the task with one means.

Until now we have been considering the case of minimal consumption of technical devices (use of the next means began only after failure of the preceding one was ascertained). In a number of cases this does not happen. For example, if the reliability of a radiosond is not great, but the time used to ascertain the fact of failure is long in comparison with the time allotted for sounding, then sondes should be launched in groups of B.

It is not difficult to show that in this case

$$M_{B\infty} = \frac{B}{1 - (1 - P)^B} \tag{13}$$

and $M_{B\infty} > M_\infty$, and $N = N_B$.

Example 6.1.1

Compare values N, M and M_B with $P = 0{,}7$, $P_N = 0{,}95$; B = 1 and 2.

Necessary reserve on the basis of (10) is

$$N = \frac{\log(1 - 0{,}95)}{\log(1 - 0{,}7)} = 2{,}5;$$

average consumption with $B = 1$ on the basis of (12) is

$$M_\infty = \frac{1}{0{,}7} = 1{,}4;$$

average consumption with B = 2 based on (13) is

$$M_{B\infty}=\frac{2}{1-(1-0,7)^2}=2,2.$$

From the example, it is evident that values of reserve N, average consumption M_∞ and $M_{B\infty}$ differ very greatly from each other

If there is the possibility of rapid maneuvering of reserves (transfer from performing one task to performing another), then with a rather large volume of tasks to be accomplished, it is possible to base calculations on average consumption without creating additional reserves. If these possibilities do not exist, then at each point where a task is accomplished, it is necessary to have a reserve, expending certain means for its content.

Failure of the radiosonde in flight can also lead to additional expenditures connected, for example, with emergency preparation of the next radiosonde, with difficulties in analyzing data in view of the time shift of measurements, etc. These expenditures are proportional to the number of failures.

Thus, the possibility of failures of single-use technical devices usually necessitates having reserves, increases the consumption of technical devices and, finally, leads to additional expenditures in proportion to the number of failures.

The mathematical expectation of expenditures for fulfilling a task in this case is

$$C=\varphi_1(M)+\varphi_2(N-M)+\varphi_3(1-P). \tag{14}$$

If reserves cannot be maneuvered, then

$$C=\frac{C_1}{P}+C_{13}\left[\frac{\log(1-P_N)}{\log(1-P)}-\frac{1}{P}\right]+C_2(1-P). \tag{15}$$

where C_1 - expenditures for one-time use of article; C_{13} — expenditures for keeping one article in reserve; C_2 - additional expenditures connected with the failure of a technical device.

In compact form this dependence can be written as follows:

$$C=\Phi(P,\ P_N,\ C_1,\ C_{13},\ C_2). \tag{16}$$

Let us consider here summary expenditures in the use of multiple-use technical devices.

Let a business operate computers, the cost of acquiring which is $C_{п}=f(\lambda)$, where λ is the intensity of failures. Assuming that failure times are distributed exponentially, we find that the probability of no-fail operation in time t is

$$P=\exp(-\lambda t); \tag{17}$$

average time between failures is

$$t_{ср}=\frac{1}{\lambda} \tag{18}$$

and the average number of failures in time t is

$$n_{ср}=t\lambda. \tag{19}$$

Assuming that the time for eliminating a failure averages $t_{вос}$, we find that effective time $t_{пол}$ is equal to

$$t_{пол}=t-t\lambda t_{вос}=t(1-\lambda t_{вос}). \tag{20}$$

Here we are assuming that failure does not cause additional loss of machine time due to loss of information concerning the accomplished part of the task, i.e., we are assuming that the accomplished part of the task is continuously (or quite frequently) printed out. Summary expenditures for one hour of machine time are

$$C_{мч}=\frac{C_{п}(\lambda)+C_{э}t}{t(1-\lambda t_{вос})}, \tag{20}$$

where $C_{э}$ — cost of an hour of machine operation, regardless of whether it is being repaired or is operating.

If the machine is used to control a production process, then its failure will cause a loss of time of equipment involved in this process and additional expenditures (penalty). In this case, spare machines may also be needed.

Without dwelling on other technical devices, we note that in all cases failures can lead to the following additional expenditures:

-- the necessity of a stock (reserve) of technical devices;

-- additional consumption of technical devices;

-- "penalties,", i.e. expenditures developing with the appearance of failures. In each individual case, based on the function specifics of the device and keeping in mind these groups of expenditures, we can write formulas connecting summary expendi-

tures for fulfilling a task with the reliability of the technical device.

We shall now formulate a general problem of determining optimal reliability and ways of providing it.

We must minimize summary expenditures

$$C = \Phi(P,\ P_N,\ C_1,\ C_{1\text{э}},\ C_2) \tag{21}$$

with limitations

$$\begin{gathered} C_1 = C(\bar{a},\ \bar{K},\ K_{\text{з}},\ \bar{C}_{\text{э}},\ \bar{C}_{\text{ох}},\ \bar{C}_{\text{з}},\ \bar{t}_{\text{т}},\ \bar{t}_{\text{з}},\ n_{\text{и}},\ N_{\text{и}}); \\ P = P(\bar{K},\ K_{\text{з}},\ \bar{C}_{\text{э}},\ \bar{C}_{\text{ох}},\ \bar{C}_{\text{з}},\ \bar{t}_{\text{т}},\ \bar{t}_{\text{з}},\ \bar{n}_{\text{и}},\ N_{\text{и}}); \end{gathered} \tag{22}$$

with constraints

$$\begin{gathered} Q_{\text{доп}} \geqslant Q(\bar{K},\ K_{\text{з}},\ \bar{C}_{\text{э}},\ \bar{C}_{\text{ох}},\ \bar{C}_{\text{з}}); \\ W_{\text{доп}} \geqslant W(\bar{K},\ K_{\text{з}},\ \bar{C}_{\text{э}},\ \bar{C}_{\text{ох}},\ \bar{C}_{\text{з}}) \end{gathered} \tag{23}$$

due to selection of $\bar{K},\ K_{\text{з}},\ \bar{C}_{\text{э}},\ \bar{C}_{\text{ох}},\ \bar{C}_{\text{з}},\ \bar{t}_{\text{т}},\ \bar{t}_{\text{з}},\ n_{\text{и}}$.

Usually this is an extremely complex problem; therefore, it is, as a rule, broken down into a number of partial problems, examples of which are described in the following paragraphs.

In conclusion, we present a very simple example of selecting optimal reliability.

Let the purchase cost of a computer be the following function of intensity of failures:

$$C_{\text{и}}(\lambda) = k_c\left(1 + \frac{k_\lambda}{\lambda}\right). \tag{24}$$

We must determine optimal reliability (in this case its characteristic is λ) if summary expenditures are determined according to formula (20). Then

$$C_{\text{уд}} = \frac{k_c + \dfrac{k_c k_\lambda}{\lambda} + C_{\text{э}}t}{t(1 - \lambda t_{\text{вос}})} = \frac{(k_c + C_{\text{э}}t)\lambda + k_c k_\lambda}{\lambda t(1 - \lambda t_{\text{вос}})}. \tag{25}$$

We find the minimum of this function

$$\frac{\partial C_{\text{уд}}}{\partial \lambda} = \frac{\lambda t(1 - \lambda t_{\text{вос}})(k_c + C_{\text{э}}t) - [(k_c + C_{\text{э}}t)\lambda + k_c k_\lambda](t - 2\lambda t t_{\text{вос}})}{[\lambda t(1 - \lambda t_{\text{вос}})]^2} = 0.$$

Evidently, a condition of fulfillment of this equality is the equality of the numerator to zero, from which

$$\lambda^2 - \lambda \frac{2k_c k_\lambda}{k_c + C_э t} - \frac{k_c k_\lambda}{t_{\text{вос}}(k_c + C_э t)} = 0;$$

$$\lambda = \frac{k_c k_\lambda}{k_c + C_э t}\left(1 + \sqrt{1 + \frac{k_c + C_э t}{t_{\text{вос}} k_c k_\lambda}}\right). \qquad (26)$$

Using this dependence, we can determine the optimal value of λ.

<u>Example 6.1.2</u>. The cost of a computer in rubles is the following function of its reliability:

$$C_{\text{в}} = 10\,000\left(1 + \frac{0.1}{\lambda}\right),$$

where λ is expressed in units per hour. The cost of operation of a machine per hour is 10 rubles. Recovery time - 3 hours. The machine is designed to operate for 30,000 hours. Determine the optimal reliability of the machine.

We have

$k_c =$ 10,000 rubles; $k_\lambda =$ 0.1 ruble/hr;

$C_э =$ 10 rubles/hr; t = 30,000 hours; $t_{\text{вос}} =$ 3 hours.

Having substituted these values in (26), we obtain

$$\lambda = \frac{10\,000 \times 0{,}1}{10\,000 + 10 \times 30\,000}\left(1 + \sqrt{1 + \frac{10\,000 + 10 \times 30\,000}{3 \times 10\,000 \times 0{,}1}}\right) = 0{,}036 \text{ failures per hour.}$$

According to (25) the cost of an hour of machine time is

$$C_{\text{вч}} = \frac{(10\,000 + 10 \times 30\,000) \times 0{,}036 + 10\,000 \times 0{,}1}{0{,}036 \times 30\,000\,(1 - 0{,}036 \times 3)} = 12{,}65 \text{ rubles.}$$

If from intuitive reasoning have higher reliability, we take λ = 0.01 failures per hour, then we obtain

$$C_{\text{вч}} = \frac{(10\,000 + 10 \times 30\,000) \times 0{,}01 + 10\,000 \times 0{,}1}{0{,}01 \times 30\,000\,(1 - 0{,}01 \times 3)} = 14{,}09 \text{ rubles.}$$

i.e., 10% higher than in the preceding case.

6.2. Determination of an Optimal Conditioning Regime, Preventive Operations and Cost of Individual Components

Let us consider the question of selecting optimal characteristics of an individual component. In the future we shall assume that optimal selection of these characteristics will also ensure optimality of the entire system as a whole.

Conditioning of components. One way of improving the reliability of a component is conditioning it, i.e. testing for a certain length of time, with subsequent installation in the technical device of those components which did not fail during conditioning.

We determine first of all the duration of conditioning $t_{\text{т}}$, after which the probability of no-fail operation of a component P(t) for given time t is maximal.

The average intensity of failures is

$$\lambda_{\text{cp}} = \frac{1}{t}\int_{t_{\text{т}}}^{t_{\text{т}}+t} \lambda(t)\,dt. \qquad (1)$$

It is obvious that minimizing λ_{cp}, we thereby maximize P(t).

Having equalized the derivative of λ_{cp} according to $t_{\text{т}}$ to zero, we obtain

$$\frac{1}{t}\left[\lambda(t+t_{\text{т}}) - \lambda(t_{\text{т}})\right] = 0,$$

from which

$$\lambda(t+t_{\text{т}}) = \lambda(t_{\text{т}}). \qquad (2)$$

This equation makes it possible to calculate $t_{\text{т}}$ with known function $\lambda(t)$.

Let us consider sufficient condition of the existence of minimum λ_{cp}. From (2) we have

$$\frac{\partial^2\lambda_{\text{cp}}}{\partial t_{\text{т}}^2} = \frac{1}{t}\left[\frac{\partial\lambda(t+t_{\text{т}})}{\partial t_{\text{т}}} - \frac{\partial\lambda(t_{\text{т}})}{\partial t_{\text{т}}}\right]. \qquad (3)$$

If function λ(t) is convex downward, then

$$\frac{\partial\lambda(t+t_т)}{\partial t_т}<0;\ \frac{\partial\lambda(t_т)}{\partial t_т}<0$$

at points where condition (2) is fulfilled. Then from (3) it follows that

$$\frac{\partial^2\lambda_{ср}}{\partial t_т^2}<0,$$

i.e., there is a minimum of $\lambda_{ср}$.

In the majority of cases λ(t) is set by a table, which can be used to construct a graph. To determine the optimal value of $t_т$ the graph can be approximated by

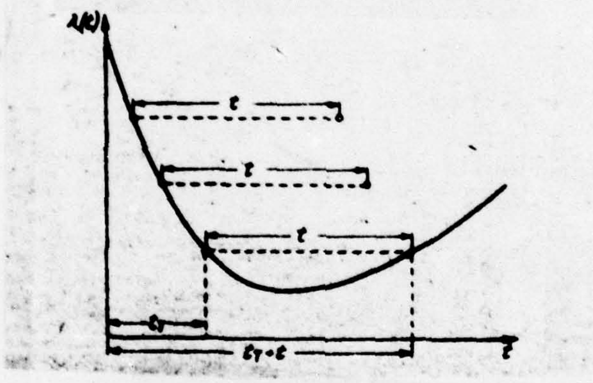

Figure 6.2.1. Determination of optimal time of conditioning a component.

some analytical function and then, using formula (2) $t_т$ determined. However, it is much simpler to do this by the graphic method (Fig. 6.2.1). For this we plot t on a straight line and move it parallel to the axis of abscissa so that its left edge touches curve λ(t). And we stop at the point where its right edge touches curve λ(t).

Analysis of this graphic method of plotting $t_т$ makes it possible to reach certain interesting conclusions immediately. If function λ(t) is monotonically increasing or monotonically decreasing, then no optimal of value $t_т$ exists, i.e. conditioning in general is unfeasible. If λ(t) contains a horizontal section whose length is greater than t, then many solutions exist (arrangement of straight line with t at

any part of this segment). However, for reasons of the feasibility of minimal time of conditioning it follows that t_T must be equal to the time at the start of the horizontal segment. If $\lambda(t)$ = const (horizontal segment begins at 0), then conditioning is not feasible.

Let us consider the question of conditioning in more detail. Let the cost of one component be C_1, the cost of one hour of conditioning $-C_T$ and loss from failure of the component C_2. The total components in the technical device -- I. In conditioning, on the average, $l_1 \exp\left[-\int_0^{t_T} \lambda(t)\,dt\right]$ components will not fail,

where l_1— number of elements being conditioned. Assuming that this number must be equal to I, we obtain

$$l_1 = l \exp\left[\int_0^{t_T} \lambda(t)\,dt\right]. \tag{4}$$

During operation of the components after conditioning, on the average

$l\left\{1 - \exp\left[-\int_{t_T}^{t_T+t} \lambda(t)\,dt\right]\right\}$ components will fail.

In this case, summary expenditures are made up of the cost of components undergoing conditioning $C_1 l_1$, the cost of conditioning $C_T l_1 t_T$ (here it is assumed that failed components are not removed during conditioning) and loss from failure of the components $C_2 l\left\{1 - \exp\left[-\int_{t_T}^{t_T+t} \lambda(t)\,dt\right]\right\}$, i.e.

$$C = l(C_1 + C_T t_T)\exp\left[\int_0^{t_T} \lambda(t)\,dt\right] + lC_2\left\{1 - \exp\left[-\int_{t_T}^{t_T+t} \lambda(t)\,dt\right]\right\}.$$

We must minimize value C:

$$\frac{\partial C}{\partial t_T} = lC_1 \exp\left[\int_0^{t_T} \lambda(t)\,dt\right]\lambda(t_T) + lC_T t_T \exp\left[\int_0^{t_T} \lambda(t)\,dt\right]\lambda(t_T) + lC_T \exp\left[\int_0^{t_T} \lambda(t)\,dt\right] + lC_2 \exp\left[-\int_{t_T}^{t_T+t} \lambda(t)\,dt\right] \times [\lambda(t_T + t) - \lambda(t_T)] = 0$$

or

$$\frac{C_1 + C_T t_T}{C_2} + \frac{C_T}{C_2}\frac{1}{\lambda(t_T)} = \exp\left[-\int_{t_0}^{t_T+t} \lambda(t)\,dt\right]\left[1 - \frac{\lambda(t_T + t)}{\lambda(t_T)}\right]. \tag{5}$$

It is most convenient to solve this equation graphically (again assuming tabular means of setting $\lambda(t)$). For this we construct a graph and the function of the right and left parts of the equation from t_T and find the point of their intersection.

Example 6.2.1. Let the intensity of failures for an electron tube be set by analytical function

$$\lambda = \frac{\lambda_0}{1 + kt},$$

i.e. the intensity of failures decreases with time.
Then

$$\frac{C_1}{C_2 kt} + \frac{C_T}{C_2 \lambda_0 kt} + t_T \left(\frac{C_T}{C_2 kt} + \frac{C_T}{C_2 \lambda_0 t} \right) = (1 + kt_T + kt)^{-\frac{\lambda_0}{k}}. \qquad (6)$$

We must find the optimal conditioning time for electron tubes t_T, if λ_o = 0.02 1/hr, k = 0.01 1/hr, time of operation of the electron tube t = 100 hours, the cost of an electron tube C_1 = 2 rubles, loss from failure C_2 = 50 rubles, cost of an hour of conditioning C_T = 0.01 ruble/hour.

Having indicated the left part of equation (6) by f_1, and the right by f_2, after substituting known values we obtain

$$f_1 = \frac{2}{50 \times 0{,}01 \times 100} + \frac{0{,}01}{50 \times 0{,}02 \times 0{,}01 \times 100} + t_T \left(\frac{0{,}01}{50 \times 0{,}01 \times 100} - \frac{0{,}01}{50 \times 0{,}02 \times 100} \right) = 0{,}05 + 0{,}0003\, t_T;$$

$$f_2 = (1 + 0{,}01 t_T + 0{,}01 \times 100)^{-\frac{0{,}02}{0{,}01} - 1} = (2 + 0{,}01 t_T)^{-3}.$$

Now we set t_T from 0 through 10 hours and we calculate f_1 and f_2 until the sign $f_1 - f_2$ changes. Results are given in Table 6.2.1

TABLE 6.2.1

t_T	0	10	20	30	40	50	60
f_1	0,050	0,053	0,056	0,059	0,062	0,065	0,068
f_2	0,125	0,108	0,094	0,082	0,072	0,064	0,057

From this it is evident that optimal $t_T \approx 50$ hours.

Preventive replacement of components. To increase the reliability of technical devices, components with comparatively low reliability can be replaced after depletion of certain resource $t_з$. It is not feasible to have a very high value of $t_з$ here, as this increases the frequency of failure during operation; with a low $t_з$, expenditures for replacement of components are increased. We shall consider a sufficiently long interval of time T in comparison with $t_з$.

Let $C_з$ — expenditures in preventive replacement of a component and C_2 — in failure during operation. Then the average cost of replacing components in time T is

$$C_{ср} = n_1 C_2 + n_2 C_з, \tag{7}$$

where n_1 — number of components failing during operation

$$n_1 = nP_1 = n\,[1 - P(t_з)]; \tag{8}$$

n_2 — number of components replaced in preventive operations,

$$n_2 = nP_2 = nP(t_з); \tag{9}$$

n — average number of replaced components

$$n = \frac{T}{t_{срз}}; \tag{10}$$

$t_{срз}$ — average lifetime of the component under the condition of replacements;

$P(t_з)$ — the probability of no-fail operation of the component in time $t_з$.

Having substituted in (7) n_1, n_2 and n from (8), (9) and (10), we obtain

$$C_{ср} = \frac{T}{t_{срз}}\{C_2\,[1 - P(t_з)] + C_з P(t_з)\}.$$

Designating $\frac{C_2}{C_з} = \gamma$ and $\frac{C_{ср} t_{ср}}{T C_з} = K$, where $t_{ср}$ — the average lifetime of a component in the absence of preventive replacements, and replacing $t_{срз}$ with its value, we obtain

$$K = \frac{t_{ср}}{\int\limits_0^{t_з} P(t)\,dt}\{P(t_з) + \gamma\,[1 - P(t_з)]\}. \tag{11}$$

The problem now consists of selecting the $t_з$, at which K reaches maximum.

We recall that $P(t) = \exp\left[-\int\limits_0^t \lambda(t)\,dt\right]$. Let the distribution law of

component life be exponential. In this case $\lambda = \text{const},\ t_{cp} = \frac{1}{\lambda}$. Then

$P(t_з) = \exp(-\lambda t_з)$ and

$$K = \frac{1}{\lambda \int\limits_0^{t_з} \exp(-\lambda t)\,dt} \{\exp(-\lambda t_з) + \gamma[1 - \exp(-\lambda t_з)]\}.$$

After uncomplicated conversions we obtain

$$K = \frac{1}{\exp(\lambda t_з) - 1} + \gamma.$$

Obviously, in this case no optimal value of $t_з$ exists (criterion K decreases monotonically with increase of $t_з$). Therefore, with exponential distribution of the lifetime of components, their preventive replacement is feasible.

Let the distribution of component lifetime follow Weibull law. Then

$$P(t) = \exp\left(-\frac{t^m}{t_0}\right);$$

$$t_{cp\,з} = \int\limits_0^{t_з} \exp\left(-\frac{t^m}{t_0}\right) dt =$$

$$= t_{cp} \frac{1}{\Gamma\left(\frac{1}{m}\right)} \int\limits_0^{\frac{t_з^m}{t_0}} z^{\frac{1}{m}-1} \exp(-z)\,dz$$

or

$$t_{cp\,з} = t_{cp} P_0\left(\frac{2t_з^m}{t_0}\right),$$

where $P_0(x)$ — distribution function χ-square with $k = 2/m$ degrees of freedom (see [54]).

Then

$$K = \frac{\exp\left(-\frac{t_з^m}{t_0}\right) + \gamma\left[1 - \exp\left(-\frac{t_з^m}{t_0}\right)\right]}{P_0\left(\frac{2t_з^m}{t_0}\right)}. \tag{12}$$

Analytical search for $t_з$, at which K is minimum is difficult in this case. However, numerical solution of this problem (in view of the necessity of finding the functional extremum of only one variable) presents no difficulties.

Example 6.2.2. The distribution of component lifetime follows Weibull's law with parameters $m = 2$ and $t_{cp} = 15$ hours. Cost of replacing a component: preventive $C_з = 2.5$ rubles, with failure during operation $C_2 = 25$ rubles. Find the optimal time for pre-

ventive replacements.

We convert equation (12), assuming that $m = 2$, and introducing designation $\frac{t_3^2}{t_0} = y$. Then we obtain

$$K = \frac{\exp(-y) + \gamma[1 - \exp(-y)]}{\Phi(4y^2)},$$

where Φ — Laplace function (it is the same as distribution function χ-square with one degree of freedom).

For a case of $m = 2$ $t_{cp} = \frac{\sqrt{\pi}}{2}\sqrt{t_0}$, from which $y = \frac{\pi}{4}\left(\frac{t_3}{t_{cp}}\right)^2$

In this example

$$\gamma = \frac{C_2}{C_1} = \frac{25}{2,5} = 10.$$

We tentatively determine the upper limit of change of y. From intuitive reasoning it is clear that it is hard to expect optimal t_3 greater than t_{cp} (i.e. to conduct preventive replacement after the component has served its average lifetime).

From that we take $y_{наиб} \approx 1$. The lease value of y is evidently 0.

We use the method of dichotomy, assuming that the function $K(y)$ is unimodal, to find the minimum of K according to y. We select the value ε based on accuracy of calculations equal to 0.01. We stipulate conducting 10 calculations of K.

In the dichotomy method, we divide the interval of indeterminacy in half, thus finding y_n. We conduct calculation at points $y_n \pm \varepsilon$ and, comparing $K(y_n + \varepsilon)$ and $K(y_n - \varepsilon)$, we determine the new interval of indeterminacy.

Results of calculations are given in Table 6.2.2.

In this case the optimal value of y is located between 0.656 and 0.687.

TABLE 6.2.2

Cycle	Interval of indeterminacy before cycle	y_n	$y_n+\epsilon$	$K(y_n+\epsilon)$	$y_n-\epsilon$	$K(y_n-\epsilon)$
1	0—1	$\frac{1-0}{2}=0{,}500$	0,510	6,5491	0,490	6,7681
2	0,5—1	$0{,}5+\frac{1-0{,}5}{2}=0{,}750$	0,760	5,9143	0,740	5,8735
3	0,5—0,75	$0{,}5+\frac{0{,}75-0{,}5}{2}=0{,}625$	0,635	5,8593	0,615	5,9062
4	0,625—0,75	$0{,}625+\frac{0{,}75-0{,}625}{2}=$ $=0{,}687$	0,697	5,8222	0,677	5,8181
5	0,625—0,687	0,656	0,666	5,8224	0,646	5,8370
6	0,656—0,687					

Having selected the average value of y_0= 0.672, we guarantee error no greater than 0.016.

Selection of the optimal cost of individual components. Let us consider the question of combining the optimal reliability of individual components.

Let the cost of each component be represented by the following function of its reliability:

$$C_i = \frac{k_i}{(1-P_i)^{a_i}} = \frac{k_i}{q_i^{a_i}}, \tag{13}$$

where k_i and a_i — coefficients; q_i — probability of failure.

In practical problems, varying k_i and a_i, we must always obtain a close agreement between approximate curves and existing statistical data.

In the absence of a reserve, the reliability of a technical device can be evaluated as follows. The probability of no-fail operation P is calculated according to formula

$$P = \prod_{i=1}^{l} p_i = \prod_{i=1}^{l}(1-q_i) \approx 1 - \sum_{i=1}^{l} q_i. \tag{14}$$

The approximate equality is valid with small values of q_i, which is usually the case in practice.

Let the required reliability of a technical device be given when the following value is also given

$$\sum_{i=1}^{l} q_i = 1 - P. \tag{15}$$

We state the problem of selecting q_i so that the summary cost of the device

$$C = \sum_{i=1}^{l} C_i = \sum_{i=1}^{l} \frac{k_i}{q_i^{\alpha_i}} \tag{16}$$

is minimal.

Having determined q_1 from (15), and having substituted it in (16), we obtain

$$C = \frac{k_1}{\left(1 - P - \sum_{i=2}^{l} q_i\right)^{\alpha_1}} + \sum_{i=2}^{l} \frac{k_i}{q_i^{\alpha_i}}. \tag{17}$$

The necessary condition of existence of min C will be equality to zero of all partial derivatives of C according to q_i. We write these conditions:

$$\frac{k_1\alpha_1}{\left(1 - P - \sum_{i=2}^{l} q_i\right)^{\alpha_1+1}} = \frac{\alpha_2 k_2}{q_2^{\alpha_2+1}};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{k_1\alpha_1}{\left(1 - P - \sum_{i=2}^{l} q_i\right)^{\alpha_1+1}} = \frac{\alpha_l k_l}{q_l^{\alpha_l+1}}.$$

Therefore,

$$\frac{\alpha_i k_i}{q_i^{\alpha_i+1}} = \mathrm{const} = A, \tag{18}$$

from which

$$q_i = \left(\frac{\alpha_i k_i}{A}\right)^{\frac{1}{\alpha_i+1}}.$$

Having substituted q_i in (14), we obtain

$$1-P=\sum_{i=1}^{l}\left(\frac{\alpha_i k_i}{A}\right)^{\frac{1}{\alpha_i+1}},$$

from which with $\alpha_i = \mathrm{const}$

$$A=\left[\frac{\sum_{i=1}^{l}(\alpha_i k_i)^{\frac{1}{\alpha_i+1}}}{1-P}\right]^{\alpha_i+1}$$

and finally

$$q_i=(1-P)\frac{\alpha_i^{\frac{1}{\alpha_i+1}}k_i^{\frac{1}{\alpha_i+1}}}{\sum_{i=1}^{l}(\alpha_i k_i)^{\frac{1}{\alpha_i+1}}}. \tag{19}$$

Using this formula, we can solve the problem of selecting optimal reliability of components if the reliability of the entire technical device is given or we can construct for this device function $P = f(C)$ with optimal distribution of the reliability of individual components. Based on this function, we can use the approach described in § 1 to select optimal reliability for the entire technical device.

We point out that the optimal conditioning regime can be selected regardless of the total reliability of a technical device, just like the regime of preventive replacement of components.

Example 6.2.3. A technical device consists of 5 components, the reliability of each of which is the following function of cost:

$$C_i=\frac{k_i}{q_i}, \tag{20}$$

where the values of k_i are given in Table 6.2.3.

TABLE 6.2.3

Number of component	1	2	3	4	5
k_i	10	20	30	40	50

TABLE 6.2.4

Number of component	1	2	3	4	5
q_i	0,024	0,034	0,042	0,048	0,052
C_i	420	522	724	836	935

The required reliability of the technical device is $P = 0.8$. Determine optimal values of q_i and corresponding C_i, at which summary expenditures are minimized. In this case $a_i = 1$.

Using formula (19), we obtain for the first component

$$q_1 = (1 - 0{,}8)\,\frac{10^{1/2}}{10^{1/2} + 20^{1/} + 30^{1/2} + 40^{1/2} + 50^{1/2}} = 0{,}024.$$

According to formula (20), we obtain

$$C_1 = \frac{10}{0{,}024} \approx 420.$$

Results of calculations of remaining q_i and C_i are given in Table 6.2.4.

6.3. Selection of Optimal Redundancy of Components. Optimal Spare Parts (ZIP)

One of the effective methods of improving the reliability of a technical device is redundancy of its components. However, redundancy also has considerable disadvantages, as it increases the number of components of which the technical device is composed, the weight and dimensions of this device, increasing its cost. Let the reliability of a technical device be given. With redundancy, the probability of no failure is

$$P = \prod_{i=1}^{l} (1 - q_i^{k_i}), \tag{1}$$

where

$$q_i = 1 - P_i. \tag{2}$$

We must determine the set of k_i (vector $\overline{K}$) which will minimize the summary cost of the technical device.

Taking into account the above, turning to formula (6.1.3), we write the following expression for the cost of the technical device:

$$C_1 = C_n + \sum_{i=1}^{l} (C_i + a_1 v_1 g_i + a_2 \mu v_2 w_i)\, k_i.$$

or, designating

$$u_i = C_i + a_2\nu_1 g_i + a_3\mu\nu_2 w_i; \tag{3}$$

$$\Delta C = C_1 - C_н, \tag{4}$$

we obtain

$$\Delta C = \sum_{i=1}^{I} u_i k_i. \tag{5}$$

We recall that there can be constraints on weight and dimensions of a technical device, which, taking into account equations (6.1.5) and (6.1.6), can be written as follows:

$$\Delta Q_{доп} \geqslant \sum_{i=1}^{I} v_i k_i; \tag{6}$$

$$\Delta W_{доп} \geqslant \sum_{i=1}^{I} w_{ni} k_i, \tag{7}$$

where

$$v_i = \nu_i g_i + \mu\nu_2 w_i; \tag{8}$$

$$w_{ni} = \nu_2 w_i. \tag{9}$$

Thus, we can formulate one of the problems of selecting optimal redundancy.

We must select vector $\overline{K}$ (with componants k_i) so that with observance of condition (1) and constraints (6) and (7) ΔC, determined by expression (5), is minimized. Usually this is a problem of nonlinear (condition (1)) integral (values of k_i must be whole numbers) programming, and there are no general effective methods of solving it.

If I is not very large, then we can use the method of complete sorting, the algorithm of which is quite simple.

Setting all possible arrangements of numbers 1, 2, ..., $k_д$ (where $k_д$ – permissible level of redundancy) according to I (where I – the number of components), for each variant we test fulfillment of conditions (6), (7) and $P < \prod_{i=1}^{I}(1 - q_i^{k_i})$ and we reject all variants in which these conditions are not met.

For remaining variants we calculate ΔC and select from them the one at which ΔC is minimal. We note that the total number of possible variants is equal to $k_{\text{п}}^{I}$ and even at I = 10 and $k_{\text{п}}=3$ it reaches almost 60,000.

Considering the simplicity of calculating the variant and the high speed of modern computers, this method can actually be used quite extensively.

Also possible are other methods, differing by greater economy.

To solve this problem we can make use of the fastest descent method. The essence of the algorithm based on this method consists of the following.

A table of values is compiled

$$
\begin{array}{c}
q_1, q_2, \ldots, q_I; \\
q_1^2, q_2^2, \ldots, q_I^2; \\
\cdots\cdots\cdots \\
q_1^{k_{\text{п}}}, q_2^{k_{\text{п}}}, \ldots, q_I^{k_{\text{п}}}.
\end{array}
$$

Based on the values of this table, using formula

$$
v_i(k_i) = \frac{q_i(k_i) - q_i(k_i+1)}{C_i\,[1 - q_i(k_i)]} \tag{10}
$$

we calculate a table of rates of change in the characteristics of reliability per unit of cost:

$$
\begin{array}{c}
v_1(1), v_2(1), \ldots, v_I(1); \\
v_1(2), v_2(2), \ldots, v_I(2); \\
\cdots\cdots\cdots\cdots \\
v_1(k_{\text{п}}), v_2(k_{\text{п}}), \ldots, v_I(k_{\text{п}}).
\end{array}
$$

Transition from the initial variant to the next occurs where the rate is fastest. For each new variant $\overline{K}$ we calculate values of P and ΔC and test fulfillment of conditions (6) and (7). Calculation shops when the required value of P is reached.

One more algorithm for solving this problem has been suggested by A. G. Korman [29]. According to this algorithm, calculation is conducted as follows.

We calculate function

$$
y_i = \frac{\Delta L_i(k_i)}{u_i} = \frac{1}{u_i} \ln\left[1 + \frac{q_i^{k_i}(1-q_i)}{1-q_i^{k_i}}\right] \tag{11}
$$

for all components.

For one component we arbitrarily set the value of k^*_i. In this component $y_i(1)$ can have the least value. We calculate $l = y_i(k^*_i)$.

For other components, values of k_i are selected from condition $y_i(k_i) = l$ and is rounded off to the nearest whole number larger or equal to 1. Thus, $\bar{K}$ is determined.

Then, using formulae (1) and (5), we calculate P and ΔC and test the fulfillment of conditions (6) and (7).

Setting different values of l, we can calculate $P(\Delta C)$ with optimal combinations of k_i and either use this function to determine optimal reliability or, if the required reliability is set, immediately determine optimal $\bar{K}$ at which the required reliability is achieved.

Finally, in the simplest case, when q_i are small and constraings (6) and (7) absent, we can obtain an analytical solution.

In this case

$$P \approx 1 - \sum_{i=1}^{l} q_i^{k_i}. \tag{12}$$

We shall use the method of Lagrangian multipliers, for which we compile function

$$F = \sum_{i=1}^{l} u_i k_i - \lambda \sum_{i=1}^{l} q_i^{k_i}. \tag{13}$$

Having converted the derivative according to k_i to zero, we obtain

$$u_i - \lambda q_i^{k_i} \ln q_i = 0, \tag{14}$$

from which

$$\frac{q_i^{k_i} \ln q_i}{u_i} = \frac{1}{\lambda}. \tag{15}$$

We designate

$$a_i = -\frac{u_i}{\ln q_i}, \tag{16}$$

then we obtain

$$q_i^{k_i} = -\frac{a_i}{\lambda}. \tag{17}$$

Having substituted (17) in equation (12), we obtain

$$1 - P = -\frac{1}{\lambda}\sum_{i=1}^{I} a_i.$$

From this we find the value of λ and having substituted it in (17), we obtain

$$q_i^{k_i} = \frac{(1-P)\,a_i}{\sum\limits_{i=1}^{I} a_i}.$$

Common logarithms can be used instead of natural k_i,

$$k_i = \frac{\ln(1-P) + \ln a_i - \ln\sum\limits_{i=1}^{I} a_i}{\ln q_i}. \tag{18}$$

If we disregard the time when a failed component is replaced by a component in the reserve system of the technical device, then the problem of selecting the optimal set of spare parts (ZIP) is very close to the one considered, and to solve it we can use the methods described above, correcting equation (5) and inequalities (6) and (7).

Example 6.3.1. Construct the dependence of the reliability of a technical device on cost with optimal redundancy, if the system consists of 5 components, whose characteristics are given in Table 6.3.1.

TABLE 6.3.1.

Number of elements	1	2	3	4	5
q_i	0,05	0,075	0,10	0,125	0,15
C_i	10	20	15	25	5

In this case we assume $C_i = u_i$. Using (16), we calculate the values of a_i. Then we set the reliability of the technical device 0.7, 0.8 and 0.9, and using formula (18), we calculate k_i. We round off the obtained values of k_i to whole numbers, we calculate P according to formula (12) and C

TABLE 6.3.2

Number of elements		1	2	3	4	5	P	C
$\log q_i$		−1,301	−1,125	−1,00	−0,904	−0,824		
a_i		3,34	7,73	6,50	12,45	2,64		
$P=0,7$	*по формуле (18)	1,16	1,02	1,22	1,04	1,96	0,628	75
	**округленное	1	1	1	1	1		
$P=0,8$	по формуле (18)	1,30	1,18	1,40	1,24	2,17	0,718	90
	округленное	1	1	2	1	2		
$P=0,9$	по формуле (18)	1,53	1,45	1,70	1,57	2,54	0,894	130
	округленное	2	1	2	2	3		

* – according to formula (18)

** – rounded

according to formula $C=\sum_{i=1}^{l} C_i$. Let us note that due to rounding off in k_i the value of P will be different from that which we set.

Results of calculations are given in Table 6.3.2.

Then we can construct a graph of $P(C)$ and approximate it by an analytical function if it is necessary to select optimal reliability of the technical device or, if this reliability is given, select optimal redundancy.

6.4. Selection of Optimal Strength of Components of Mechanical Structures

The question of reliability of mechanical structures has not been discussed for a long time. As a measure of reliability (far from always objective), the practice has been to use the margin of safety, by which is understood the ratio of the supporting power of a structure to operational loads. Calculated safety margin contains errors in methods of calculating supporting power and operational loads and a margin for dispersion of these values. The validity of the safety

margin was determined from the experience of operating analogous structures; therefore, it is far from always possible to verify that structures with a high margin of safety are more reliable than a structure with a lower margin of safety, especially if we are talking about different structures, operating under different conditions.

Recently the probability approach to calculating mechanical structures has become increasingly widespread. The reliability of the operation of these structures is being evaluated according to criteria from the theory of reliability and the very concept of "margin of safety" is acquiring a new meaning.

Everything in § 6.1-6.3 is also directly related to the reliability of mechanical structures; however, in this case definite characteristics arise, on which we shall dwell in more detail.

Let us consider first of all the question of the strength calculation of some component of a mechanical structure. The component must usually satisfy two requirements:

1) it must not fail under actual operational loads (under given conditions of operation);

2) deformations must not exceed given limits.

In accordance with this we must consider two limiting states of the structure:

— in strength (structure fails);

— in deformation (structure receives inadmissible deformations).

We shall distinguish operational loads acting on the structure ($Э$) and the supporting power of the structure by strength (breaking load)(R) and by deformation (Q).

Both supporting power and operational loads are random values.

Factors causing dispersion of supporting power are:

— dispersion of characteristics of the mechanical properties of the material of the structural element;

— dispersion of the geometric dimensions of the structural element.

We note that the coefficient of variation of the limit of strength for steel varies from 2 to 6%, for titanium alloys — from 2 to 4%, for aluminum alloys — from 3 to 7%, for fiberglass — from 10 to 30% [37].

Variations of the geometric domensions of an article are determined by the class of accuracy of manufacture.

Variations of operational loads depend on the conditions under which the article is used, and can reach significant levels.

Expressions for the calculation of supporting power by strength and by deformation are distinguished, depending on kind of load; however, usually we can write

$$R=\sigma\prod_{i=1}^{l}x_i^{n_i}; \tag{1}$$

$$Q=E\prod_{i=1}^{l}x_i^{m_i}, \tag{2}$$

where x_i — geometric dimensions of structural elements (or their sums); E — elastic modulus; σ — permissible stress.

As seen from formulae (1) and (2), supporting power is a derivative of random values (sometimes up to 10); therefore, strict study of the distribution law of supporting power is connected with certain difficulties. The usual method of determining this law in important cases can be statistical testing, by which distribution density of supporting power is obtained for each specific case

$$\varphi(R)=\varphi(R, M, K); \tag{3}$$

$$\varphi(Q)=\varphi(Q, M, K), \tag{4}$$

where M — material from which a part is made. It is characterized by the distribution law of σ and E; K — quality of manufacture, characterized by the distribution law of dimensions x_i.

However, in the majority of cases encountered in practice, the distribution law of supporting power is assumed to be normal, and its parameters (mathematical

expectation and standard deviation) are found by an approximate method.

As the simplest, although approximate, method, we can recommend the method of reduced perturbations, the algorithm of which consists of the following.

We calculate

$$R_0 = a_0 \prod_{i=1}^{l} x_{i_0}^{n_i}, \tag{5}$$

where the index "0" designated mathematical expectations;

$$\sigma_R = B\,[R_0 - R(a_0 \pm \sigma_a;\ x_{10} \pm \sigma_{x1};\ \dots;\ x_{l0} \pm \sigma_{xl})], \tag{6}$$

here signs with regard to σ are taken so that σ_R decreases. The value of B is

$$B = \frac{\sqrt{1 + \frac{\pi}{2} l}}{1 + l}. \tag{7}$$

The method is based on study of the distribution of random values

$$B = \frac{\sqrt{\sum_{i=1}^{l} x_i^2}}{\sum_{i=1}^{l} x_i} \tag{8}$$

and on the hypothesis of the distribution of values of x_i as a model of normally distributed values.

Another approximate method of determining x_i can be the method of partial derivatives, according to which

$$\sigma_R \simeq \sqrt{\left(\frac{\partial R}{\partial a}\sigma_a\right)^2 + \sum_{i=1}^{l}\left(\frac{\partial R}{\partial x_i}\sigma_i\right)^2}. \tag{9}$$

Usually the criterion of the mechanical strength of a structural element is the probability of its no-fail operation, i.e. absence of both limiting states,

$$P = \text{Вер}\,(R - Э > 0;\ Q - Э > 0) = \\ = \int_0^\infty \int_0^\infty \varphi(R - Э;\ Q - Э)\, d(R - Э)\, d(Q - Э). \tag{10}$$

The reliability of a structural element only according to strength is characterized by the probability

$$P = \text{Вер}(R - Э > 0).$$

In those cases when R and $Э$ can be considered distributed according to normal law, their difference is also distributed according to normal law

$$P = \int_0^\infty \frac{1}{\sigma_{R-Э}\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{Э - R - Э_0 + R_0}{\sigma_{R-Э}}\right)\right] d(R-Э) = \\ = F_0\left[\frac{R_0 - Э_0}{\sqrt{\sigma_R^2 + \sigma_Э^2 - 2\rho_{RЭ}\sigma_R\sigma_Э}}\right], \tag{11}$$

where R_0 and $Э_0$ — mathematical expectations of supporting power and operational load; σ_R and $\sigma_Э$ — standard deviation of these values; $\rho_{RЭ}$ — coefficient of coordination between them;

$$F_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}x^2\right) dx.$$

If coefficient $\rho_{RЭ}$ is equal to zero, then equation (11) can be written in the following form:

$$P = F_0\left(\frac{\eta_0 - 1}{\sqrt{\eta_0^2 v_R^2 + v_Э^2}}\right), \tag{12}$$

where $\eta_0 = \frac{R_0}{Э_0}$;

$$v_R = \frac{\sigma_R}{R_0} \text{ and } v_Э = \frac{\sigma_Э}{Э_0}; \tag{13}$$

η_0 — relative margin of safety; v_R and $v_Э$ — coefficients of variation of loads.

Let us point out that in this case by relative margin of safety is meant the ratio of the mathematical expectation of supporting power to the mathematical expec-

tation of operational load.

Equation (12) makes it possible, with the above assumptions, to determine the probability of no-fail operation of a structural element as a function of η_0; v_R and $v_Э$.

Let us consider now the question of selecting the optimal reliability of a mechanical structure. Let the cost of expenditures for fulfilling tasks, in an analogy with (6.1.5), be determined according to formula

$$C=\frac{C_1}{P}+C_2(1-P). \tag{14}$$

The cost of a technical device, in an analogy with (6.1.4) — according to formula

$$C_1=\bar{C}(\bar{a},\ \sigma,\ \bar{X}), \tag{15}$$

where $\overline{X}$ — vector of geometric dimensions with components

$$x_1,\ x_2,\ \dots,\ x_i,\ \dots,\ x_l.$$

Let $Э_0,\ v_Э,\ v_R$ be given and, finally, an equation of type (5).

Then the problem of selecting the optimal reliability of a mechanical structure can be formulated as follows.

Select σ and $\overline{X}$ so that C (14) is minimized with connections (5), (12) and (15). Depending on the kind of specific dependences, this problem can be solved by one of the methods of mathematical programming.

Let us discuss a very simple specific example illustrating this method.

Let an airplane engine contain a cylinder of pressurized compressed air with diameter D. The mathematical expectation of pressure is $Э$, and standard deviation $\sigma_Э$. Failure of the cylinder will necessitate a forced landing; expenditures caused by the landing are indicated by C_2.

The cost connected with installing the cylinder (C_1) is a function of the thickness of its wall δ (it is determined by the loss of load capacity and cost of the cylinder itself)

$$C_1 = k_{C_1}\delta. \tag{16}$$

In this case

$$R| = \frac{2\sigma\delta}{D}. \tag{17}$$

Let v_σ be given, $v_\delta = v_D = 0$. Then $v_R = v_\sigma$.

For this case

$$P = F_0\left(\frac{\eta_0 - 1}{\sqrt{\eta_0^2 v_R^2 + v_Э^2}}\right);$$
$$\eta_0 = \frac{2\sigma\delta}{DЭ_0},$$

from which

$$\delta = \frac{DЭ_0}{2\sigma}\eta_0; \tag{18}$$
$$C_1 = \frac{k_{C_1}DЭ_0}{2\sigma}\eta_0;$$

$$C = \frac{k_{C_1}DЭ_0}{2\sigma}\eta_0 + C_2\left[1 - F_0\left(\frac{\eta_0 - 1}{\sqrt{\eta_0^2 v_R^2 + v_Э^2}}\right)\right]. \tag{19}$$

We must determine the η_0 at which C reaches minimum. In this case η_0 is easy to determine using any numerical method of search for extremum.

Example 6.4.1. Determine the optimal value of the margin of safety for the problem described above if $\frac{k_{C_1}DЭ_0}{2\sigma} = 100$; $C_2 = 10,000$; $v_R = 0{,}10$; $v_Э = 0{,}10$ and a failure will lead to an accident. Setting various values of η_0, we find C. Calculation is given in Table 6.4.1.

As is seen from Table 6.4.1, the optimal reserve of strength η_0=1.6, and the reliability corresponding to it is 0.9993.

Example 6.4.2. Determine the optimal margin of safety and reliability if breaking the cylinder does not lead to an accident, and all other characteristics are analogous to the preceeding example.

TABLE 6.4.1

η_0	1,0	1,1	1,2	1,3	1,4	1,5	1,6	1,7
$C_1 = \frac{k_{C_1} D \mathit{Э}_0}{2a} \eta_0$	100	110	120	130	140	150	160	170
$\eta_0 - 1$	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7
$\eta_0^2 v_R^2$	0,0100	0,0121	0,0144	0,0169	0,0196	0,0225	0,0256	0,0289
$\eta_0^2 v_R^2 + v_{\mathit{Э}}^2$	0,0200	0,0221	0,0244	0,0269	0,0296	0,0325	0,0356	0,0389
$\sqrt{\eta_0^2 v_R^2 + v_{\mathit{Э}}^2}$	0,112	0,149	0,156	0,164	0,172	0,180	0,188	0,197
$x = \frac{\eta_0 - 1}{\sqrt{\eta_0^2 v_R^2 + v_{\mathit{Э}}^2}}$	0,00	0,67	1,28	1,83	2,33	2,78	3,19	3,55
$P = F_0(x)$	0,500	0,749	0,900	0,9664	0,9901	0,9973	0,9993	0,9998
$1 - F_0(x)$	0,500	0,251	0,100	0,0336	0,0099	0,0027	0,0007	0,0002
$y = C_2[1 - F_0(x)]$	5000	2510	1000	336	99	27	7	2
$C = C_1 + y$	5100	2620	1120	466	239	177	167	172

Under these conditions, summary expenditures are

$$C \approx \frac{C_1}{P} = \frac{k_{C_1} D \mathit{Э}_0}{2a} \frac{\eta_0}{P}.$$

Using data of the preceeding example, we obtain the results given in Table 6.4.2.

TABLE 6.4.2

η_0	1,0	1,1	1,2	1,3
P	0,500	0,749	0,900	0,9664
C	200	146,9	133,3	134,5

As seen from Table 6.4.2, in this case optimal requirements for margin of safety (η_0 = 1.2) and reliability (P = 0.900) are much lower than in the preceeding example.

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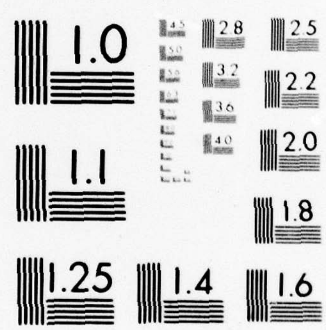
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6.5. Selection of the Optimal Order of Further Improvement of Technical Devices

When the first test model of a new technical device is designed and produced, it proceeds to testing. During tests are revealed its inadequacies, including inadequate reliability of its individual units, assemblies and parts. These inadequacies are eliminated in succeeding models by design and technological improvements, as a result of which the reliability of the succeeding model becomes higher than the one preceeding. Thus, as the manufacture and testing of models of technical devices advance, we can expect increased reliability; this increase must be a function of the number of test models (under the condition that revealed defects are eliminated in succeeding models). This is observed in practice. The question arises of the order in which to organize tests of articles to maximize the number of suitable articles in operation. Below we consider solution of this problem for a case when the model of a technical device is destroyed during testing.

<u>Determination of change in the reliability of technical devices as they are tested</u>. Let us consider a very simple model for determining the reliability of technical devices. Let it consist of N_6 blocks, the probability of no-fail operation of each of which is P_i, and the probability of failure q_i. Failures of the blocks are independent. Then reliability of the entire article before testing is

$$P_\circ = \prod_{i=1}^{N_6} P_i = \prod_{i=1}^{N_6} (1 - q_i) \approx 1 - \sum_{i=1}^{N_6} q_i. \qquad (1)$$

The probability of eliminating the cause of failure after its appearance in tests is g_i. The probability of eliminating the failure of i-block after one test is $g_i q_i$. The probability that failure will not be eliminated after n_n tests is

and

$$Q_i = (1 - q_i g_i)^{n_n} \approx \exp(-q_i g_i n_n). \qquad (2)$$

The probability of failure of i-block after n_n tests is

$$q_{i_n} = q_i \exp(-q_i g_i n_n). \qquad (3)$$

The probability of no-fail operation of i-block after n_n tests is

$$P_{i_n} = 1 - q_i \exp(-q_i g_i n_n). \qquad (4)$$

The reliability of the entire article after n_u tests is

$$P_n = \prod_{i=1}^{N_6} [1 - q_i \exp(-q_i g_i n_u)] \sim 1 - \sum_{i=1}^{N_6} q_i \exp(-q_i g_i n_u). \tag{5}$$

Let us consider a very simple case, when $q_i = \text{const} = q$ and $g_i = \text{const} = g$.

Then

$$P_0 = 1 - N_6 q;$$

$$P_n = 1 - N_6 q \exp(-q g n_u) = 1 - (1 - P_0) \exp(-q g n_u).$$

Designating $qg = k$, we obtain

$$P_n = 1 - (1 - P_0) \exp(-k n_u). \tag{6}$$

If P_i (and q_i) are not equal to each other (this is usually the case), then arranging q_i in descending order, we can approximate the obtained sequence by geometric progression

$$q_i = q_1 \alpha^{i-1}. \tag{7}$$

Then

$$P_0 = 1 - \sum_{i=1}^{N_6} q_n = 1 - \frac{q_1 (\alpha^{N_6} - 1)}{\alpha - 1}. \tag{8}$$

With insufficiently large N_6

$$P_0 \approx 1 - \frac{q_1}{1 - \alpha}, \tag{9}$$

from which we obtain the connection between α and q_i in terms of P_0:

$$q_1 = (1 - P_0)(1 - \alpha). \tag{10}$$

We determine the value of P_n, in this case, assuming $g_i = \text{const} = g$,

$$P_n = 1 - \sum_{i=1}^{N_6} q_i \exp(-g q_i n_u) = 1 - q_1 \sum_{i=1}^{N_6} \alpha^{i-1} \times$$

$$\times \exp(-g n_u q_1 \alpha^{i-1})$$

or

$$P_n = 1 - (1 - P_0) f_p [\alpha; g (1 - P_0) n_u]. \tag{11}$$

In those cases when α is close to 1 (which indicates large $N_б$), we can proceed from summation to integration.

Then we obtain

$$P_n = 1 - q_1 \int_0^{\infty} \alpha^i \exp(-b\alpha^i)\, di = 1 - q_1 \frac{1-\exp(-b)}{b \ln \alpha}, \tag{12}$$

where

$$b = g n_н q_1. \tag{13}$$

Taking into account the above

$$\begin{aligned} P_n &= 1 - (1 - P_0) \frac{1-\exp(-b)}{b} = \\ &= 1 - (1 - P_0) \frac{1-\exp(-k_1 n_1)}{k_1 n_н}, \end{aligned} \tag{14}$$

where

$$g q_1 = k_1. \tag{15}$$

Comparative Table 6.5.1 of values of $\varphi_1(kn_н) = \exp(-kn_н)$ and $\varphi_2(k_1 n_н) = \frac{1-\exp(-k_1 n_н)}{k_1 n_н}$ is given below.

TABLE 6.5.1.

$kn_н$	0,0	0,2	0,4	0,6	0,8	1,0	2,0	3,0	4,0	5,0
$\varphi_1(kn_н)$	1,0	0,819	0,670	0,549	0,449	0,368	0,135	0,050	0,018	0,007
$\varphi_2(k_1 n_н)$	1,0	0,905	0,825	0,752	0,689	0,632	0,433	0,317	0,245	0,199

As can be seen from Table 6.5.1, with uneven distribution of the intensity of failures among units of the technical device, the rate of elimination of failures decreases significantly.

Comparison of the two obtained formulae (6) and (14) with test data, as would be expected, showed closer similarity with the second. Values of k_1 vary for different articles within comparatively narrow limits: 0.013-0.034; this value is a

function of the "complexity" of the technical device.

Selection of the optimal number of tested articles. Above we determined the functional connection between the reliability of an article and the number of tested articles. We write this in general form:

$$P=P(n_{\text{и}}).$$

The number of suitable articles (without taking into account their storage time) can be determined from the following dependence:

$$M=\int_0^T P(n_{\text{и}})\left(\frac{dN_{\text{и}}}{dt}-\frac{dn_{\text{и}}}{dt}\right)dt, \tag{16}$$

where $\frac{dN_{\text{и}}}{dt}$ — production of articles per unit of time; $\frac{dn_{\text{и}}}{dt}$ — consumption of articles for testing per unit of time; T — interval of time during which given articles are produced.

If we disregard additional expenditures for tests, then the problem of determining the optimal number of tested articles can be formulated mathematically as follows: determine $\frac{dn_{\text{и}}}{dt}$ with $P(n_{\text{и}}), \frac{dN_{\text{и}}}{dt}$, given, T so that M reaches maximum. Taking the above into account

$$M=\int_0^T P\left(\int_0^t \frac{dn_{\text{и}}}{dt}dt\right)\left(\frac{dN_{\text{и}}}{dt}-\frac{dn_{\text{и}}}{dt}\right)dt. \tag{17}$$

We shall solve the problem, assuming that production possibilities are limited. Then

$$\frac{dN_{\text{и}}}{dt}=N'_{\text{и}}=f(N_{\text{и}}). \tag{18}$$

The intensity of sampling for testing in this case is the equation

$$\frac{dn_{\text{и}}}{dt}=n'_{\text{и}}=u. \tag{19}$$

We must determine u, ensuring

$$\max\int_0^T P(n_{\text{и}})[f(N_{\text{и}})-u]\,dt \tag{20}$$

with fulfillment of condition

$$f(N_{\text{и}})\geqslant u. \tag{21}$$

We use the principle of maximization of L. S. Pontryagin. The Hamiltonian can be written in the form

$$H=\psi u+P(n_n)[f(N_n)-u]. \tag{22}$$

We write the equation for generalized impulse ψ in the form

$$\dot{\psi}=\frac{\partial H}{\partial n_n}=-\frac{\partial P(n_n)}{\partial n_n}[f(N_n)-u]. \tag{23}$$

Control u enters the Hamiltonian linearly, in connection with which the maximum of H according to control u is achieved with fulfillment of conditions

$$u=\begin{cases}u_{max} \text{ при } v=\psi-P(n_n)>0;\\ u_{min} \text{ при } v=\psi-P(n_n)<0.\end{cases} \tag{24}$$

The value of u_{max} is determined by condition (21), while the value of u_{min} from condition $\frac{dn_r}{dt}\geqslant 0$ in each running moment of time.

Because

$$\frac{\partial P(n_n)}{\partial n_n}>0$$

and condition (21) exists, from equations (23) and (18) it follows that $\dot{\psi}\leqslant 0$, therefore ψ is a decreasing function. On the other hand, P — monotonic function of n_n, while $n_n=\int u dt$ — nondecreasing function.

Therefore, the condition of reversing control (24) can change the sign not more than once. As $P(n_n)>0$ and ψ decreases with the course of time, the sign can change only from positive to negative.

Thus, usually the structure of optimal control is the following: first, all articles which are produced must be tested, and then, in accordance with conditions (24), tests are to be stopped. However, this order of tests cannot be adopted in practice, as production conditions change with the course of time.

Let us consider the question of selecting the optimal number of test articles if all produced articles until a certain number are tested, and then tests are discontinued until the end of production. In this case the number of suitable articles produced in the entire time is

$$M=P(n_0)(N_n-n_0) \tag{25}$$

or

$$M = \left[1 - (1 - P_0)\frac{1 - \exp(-k_1 n_0)}{k_1 n_0}\right](N_n - n_0).$$

Designating

$$k_1 n_0 = x;$$
$$k_1 N_n = b,$$

we obtain

$$k_1 M = \left[1 - (1 - P_0)\frac{1 - \exp(-x)}{x}\right](b - x).$$

Equalizing the partial derivative of $k_1 M$ according to x to zero, after several uncomplicated conversions we obtain

$$1 - \frac{b}{x} + \frac{b\,[\exp(x) - 1]}{x^2} = \frac{\exp(x)}{1 - P_0} - \tag{26}$$

a transcendental equation with two parameters b and $(1 - P_0)$, from which we must calculate x.

To facilitate computations, we compile Table 6.5.2, in which are arranged values of β:

$$\beta = \frac{x}{b} = \frac{n_0}{N_n} = F_1(P_0; b).$$

TABLE 6.5.2

$$\beta = \frac{x}{b} \cdot 10^2$$

b	P_0								
	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
0,3	13								
0,4	20								
0,5	26								
0,6	28	8							
0,7	30	14							
0,8	32	18							
0,9	33	20	4						
1,0	34	22	8						
2,0	35	28	20	11					
3,0	34	30	24	17	9				
4,0	33	29	25	20	13	6			
5,0	32	28	25	20	15	9			
6,0	31	27	24	20	16	11	4		
7,0	30	27	24	20	16	12	6		
8,0	28	26	24	20	16	12	7		
9,0	27	25	23	20	16	13	8	1	
10	26	24	22	20	16	13	9	3	
20	21	20	18	16	15	12	9	5	1
30	17	16	15	14	13	11	9	6	2
40	15	14	13	12	11	10	9	6	3
50	13	13	12	11	10	9	8	5	3
100	10	9	8	8	7	6	6	5	3
500	4	4	4	4	3	3	2	2	1
1000	3	3	3	2	2	2	2	1	1

Note. with $b \leq \frac{2P_0}{1 - P_0}$ $\beta = 0$. - i.e. tests to increase reliability are not feasible.

In this case

$$k_1 M = bF_2(P_0; b).$$

For $\frac{k_1 M}{b} = \frac{M}{N_n} = F_2(P_0; b)$ Table 6.5.3 is compiled.

TABLE 6.5.3

$\frac{k_1 M}{b} \cdot 10^2$

b	P_0								
	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
0,3	10								
0,4	11								
0,5	12								
0,6	12	20							
0,7	13	20							
0,8	14	21							
0,9	15	21	30						
1,0	16	22	30						
2,0	23	28	34	41					
3,0	29	33	38	44	51				
4,0	34	37	42	47	53	60			
5,0	38	41	45	50	55	62			
6,0	41	44	48	52	57	63	70		
7,0	44	47	50	54	59	64	71		
8,0	46	49	52	56	61	65	72		
9,0	48	51	54	58	63	66	72	80	
10,0	50	53	56	59	64	67	73	80	
20	62	64	66	69	71	74	78	83	90
30	68	70	72	74	76	79	81	85	91
40	72	74	75	77	79	82	84	87	91
50	75	77	78	79	81	83	85	88	92
100	82	83	84	85	87	88	89	91	94
500	92	92	93	93	94	94	95	96	97
1000	94	94	95	95	96	96	97	97	98

With $x > 3$ formulae are simplified:

$$k_1 M \approx \left[1 - (1 - P_0)\frac{1}{x}\right](b - x);$$

$$\frac{\partial (k_1 M)}{\partial x} = \frac{b(1 - P_0)}{x^2} - 1 = 0,$$

from which

$$x \approx \sqrt{b(1 - P_0)}. \tag{27}$$

In this case

$$k_1 M = b\left(1 - \sqrt{\frac{1 - P_0}{b}}\right)^2. \tag{28}$$

When x is small, we obtain

$$k_1 M \approx \left[1-(1-P_0)\frac{1-\left[1-x+\frac{x^2}{2}\right]}{x}\right](b-x).$$

Selecting a partial derivative of $k_1 M$ according to x and equalizing it to zero, after uncomplicated operations we obtain

$$x=\frac{b}{2}+1-\frac{1}{1-P_0}. \qquad (29)$$

In its physical meaning, x cannot be negative. Therefore, under certain b, this value is equal to zero, indicating the nonfeasibility of tests,

$$b<\frac{2P_0}{1-P_0}. \qquad (30)$$

Let us now consider a case when a certain portion of produced articles is tested $\beta=\frac{n_u}{N_n}$ continuously, as production continues, which is often the case in practice. Then

$$M=\int_0^{N_n} P(\beta N_n)(1-\beta)\,dN_n=$$

$$=(1-\beta)\int_0^{N_n}\left[1-(1-P_0)\frac{1-\exp(-k_1\beta N_n)}{k_1\beta N_n}\right]dN_n. \qquad (31)$$

We proceed to a new variable $x=k_1\beta N_n=k_1 n_u$. Then

$$M=\frac{1-\beta}{k_1\beta}\int_0^x\left[1-(1-P_0)\frac{1-\exp(-x)}{x}\right]dx \qquad (32)$$

or

$$k_1 M=\frac{1-\beta}{\beta}-\frac{1-\beta}{\beta}(1-P_0)\left[\int_0^x\frac{dx}{x}-\int_0^x\frac{\exp(-x)}{x}\,dx\right].$$

The second integral has no primitive. To calculate it we can use series

$$\int_0^x\frac{\exp(-x)}{x}\,dx=\ln x-\frac{x}{1.1!}+\frac{x^2}{2.2!}-\frac{x^3}{3.3!}\ldots=$$

$$=\ln x-\varphi(x).$$

We make substitutions in the formula for $k_1 M$:

$$k_1 M=b(1-\beta)-\frac{1-\beta}{\beta}\varphi(b,\beta). \qquad (33)$$

TABLE 6.5.4

$$\beta = \frac{x}{b} \cdot 10^2$$

b	P_0								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0,5	2								
0,6	4								
0,7	15								
0,8	20								
0,9	24								
1,0	25								
2,0	34	22	8						
3,0	35	28	19	6					
4,0	34	29	23	14	1				
5,0	34	29	23	17	8				
6,0	33	28	24	18	11				
7,0	33	28	24	18	12	3			
8,0	32	28	24	18	12	6			
9,0	31	27	23	18	13	7			
10	31	27	23	18	13	8	1		
20	26	22	20	16	13	10	6	2	
30	23	20	17	15	12	10	7	4	
40	21	18	16	14	12	9	7	4	1
50	20	17	15	13	11	9	7	5	2
100	15	14	12	11	10	8	7	5	3
500	8	8	7	6	6	5	4	3	2
1000	6	5	5	4	4	4	3	2	1

TABLE 6.5.5

$$\frac{k_1 M}{b} \cdot 10^2$$

b	P_0								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0,5	10								
0,6	10								
0,7	10								
0,8	10								
0,9	11								
1,0	11								
2,0	16	24	30						
3,0	20	26	33	40					
4,0	24	29	35	42	50				
5,0	26	31	36	43	50				
6,0	28	32	38	44	51				
7,0	29	32	40	45	52	60			
8,0	31	34	41	46	53	60			
9,0	32	36	42	47	54	61	70		
10	34	39	43	48	54	61	70		
20	42	50	54	58	62	68	74	81	
30	53	56	58	63	67	71	76	82	
40	58	60	62	66	70	74	78	84	90
50	60	62	64	68	72	75	78	84	91
100	68	70	72	75	77	80	83	87	92
500	82	83	84	86	87	89	90	92	95
1000	86	88	89	90	91	32	93	94	96

We must find the extremum of this function. On the basis of this equation we compile Table 6.5.4, in which are arranged values of $\beta = F_3(P_0, b)$, and Table 6.5.5, in which are arranged values of

$$\frac{k_1 M}{b} = F_4(P_0, b).$$

If x is small, then we can find an analytical expression.

For this we use approximate equality

$$\exp(-x) \cong 1 - x + \frac{x^2}{2} - \frac{x^3}{6}.$$

Then

$$k_1 M \cong \frac{1-\beta}{\beta} \int_0^x \left[1 - (1 - P_0)\left(1 - \frac{x}{2} + \frac{x^2}{6}\right)\right] dx.$$

After uncomplicated conversions we obtain

$$k_1 M \cong (b - x)\left[P_0 + (1 - P_0)\frac{x}{4} - (1 - P_0)\frac{x^2}{18}\right].$$

Equalizing $\frac{\partial(k_1 M)}{\partial x}$ to zero, we find

$$x^2 - x(1 - P_0)\left(\frac{2}{3}b + 3\right) + \frac{3[b(1 - P_0) - 4P_0]}{2} = 0, \tag{34}$$

from which, equalizing x to zero, we find

$$b(1 - P_0) - 4P_0 = 0$$

or

$$b = \frac{4P_0}{1 - P_0}. \tag{35}$$

This is the condition of the start of tests. Comparing it with (30), we find that in the case of continuous tests, the value of b at which tests are not feasible is two times higher than for tests at the initial stage only.

Solving equation (34) we obtain

$$x = \frac{b}{3} + \frac{3}{2} - \sqrt{\left(\frac{b}{3} + \frac{3}{2}\right)^2 - \frac{3b}{2} + \frac{6P_0}{1 - P_0}}. \tag{36}$$

In practical calculations, the use of Tables 6.5.4 and 6.5.5 is recommended.

Example 6.5.1. For a technical device in the initial stage of testing $P_0 = 0.6$ and $k_1 = 0.02$ are determined. Determine the optimal volume of tests if they are conducted only in the initial stage and if they are continuous, if a total production of 1000 technical devices is assumed.

We calculate

$$b = k_1 N_{\text{и}} = 0{,}02 \times 1000 = 20.$$

For the first case, using Table 6.5.2, we find

$$\beta = 0.12,$$

from which $n_0 = \beta N_{\text{и}} = 0{,}12 \times 1000 = 120.$

With the aid of Table 6.5.3 we find

$$\frac{k_1 M}{b} = 0{,}74,$$

from which the number of suitable articles is

$$M = \frac{k_1 M}{b} N_{\text{и}} = 0{,}74 \cdot 1000 = 740.$$

In the second case, using Tables 6.5.4 and 6.5.5, we find

$$\beta = 0{,}10; \quad n_0 = \beta N_{\text{и}} = 0{,}10 \cdot 1000 = 100;$$

$$\frac{k_1 M}{b} = 0{,}68; \quad M = \frac{k_1 M}{b} N_{\text{и}} = 0{,}68 \cdot 1000 = 680.$$

As results of the calculations show, the second case is less favorable than the first.

7. THE USE OF OPERATIONS RESEARCH METHODS IN THE DEVELOPMENT OF TECHNICAL DEVICES

7.1. Distribution Problems

Problems of the distribution of limited resources very frequently arise in designing technical devices. In Chapter 6 we have already discussed such problems, in particular the problem of selecting optimal reliability of individual components.

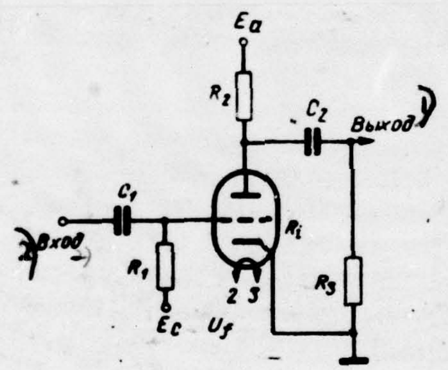

Figure 7.1.1. Circuit of low-frequency amplifier.

1 — input   2 — output

The problem of optimal distribution of margin tolerances of components of a technical device, margins of safety of individual components, etc., often come to this problem also.

Below we discuss three distribution problems: selection of optimal tolerances (i.e. distribution of expenditures among parts of a device); selection of the optimal diameter of a pipe (i.e. distribution of expenditures between capital and operational); and selection of the optimal combination of material, production accuracy

and operational loads.

<u>Determination of optimal tolerances of parts of a low-frequency amplifier</u>. For a low-frequency amplifier (Fig. 7.1.1), given are permissible deviations of the amplification factor $k_0(\Delta k_0)$, values of the distortion factor at low ($\omega_н$) frequency $M_{н\,доп}$ and at high ($\omega_в$) frequency $M_{в\,доп}$.

We must select tolerances for components of the circuit at which the cost of the amplifier will be minimized.

We shall not derive formulae connecting deviations of the amplification factor and distortion factors with deviations of characteristics of parts of the amplifier and supply voltages; we shall present only concluding formulae:

$$\frac{\Delta k_0}{k_0}=\frac{\Delta S}{S}+R_{экв}\left(\frac{1}{R_i}\frac{\Delta R_i}{R_i}+\frac{1}{R_3}\frac{\Delta R_3}{R_3}\right)+$$
$$+\left(\frac{R_{экв}}{R_2}+a\right)\frac{\Delta R_2}{R_2}+b\frac{\Delta E_a}{E_a}+c\left|\frac{\Delta E_c}{E_c}\right|+d\frac{\Delta U_f}{U_f};$$
$$M_н=-\frac{1}{(\omega_н C_1R_1)^2+1}\left(\frac{\Delta C_1}{C_1}+\frac{\Delta R_1}{R_1}\right)-$$
$$-\frac{1}{(\omega_н C_2R_3)^2+1}\left(\frac{\Delta C_2}{C_2}+\frac{\Delta R_3}{R_3}\right);$$
$$M_в=\frac{\omega_в C_0R_{экв}}{(\omega_в C_0R_{экв})^2+1}\left[\frac{\Delta C_0}{C_0}+\frac{1}{R_{экв}}\left(\frac{1}{R_i}\frac{\Delta R_i}{R_i}+\right.\right.$$
$$\left.+\frac{1}{R_3}\frac{\Delta R_3}{R_3}\right)+\left(\frac{R_{экв}}{R_2}+e\right)\frac{\Delta R_2}{R_2}+$$
$$\left.+k\frac{\Delta E_a}{E_a}+m\left|\frac{\Delta E_c}{E_c}\right|+n\frac{\Delta U_f}{U_f}\right].$$

where

$$R_{экв}=\frac{1}{\frac{1}{R_i}+\frac{1}{R_3}+\frac{1}{R_3}};$$

S — static transconductance of grid plate characteristics; a, b, c, d, e, k, m, n — coefficients, usually determined by test.

We note that the spread of supply voltages and grid plate characteristics have a very great influence on values of $\Delta k, M_{н}$ and $M_{в}$.

If we consider these parameters as given, then equations are simplified and we shall use, taking into account given constraints, the following kind:

$$\frac{1}{R_{\text{экв}}}\left(\frac{1}{R_2}\frac{\Delta R_2}{R_2}+\frac{1}{R_3}\frac{\Delta R_3}{R_3}\right)<\frac{\Delta k_0}{k_0}; \tag{1}$$

$$\frac{1}{(\omega_{н}C_1R_1)^2+1}\left(\frac{\Delta C_1}{C_1}+\frac{\Delta R_1}{R_1}\right)+\frac{1}{(\omega_{н}C_2R_3)^2+1}\times$$
$$\times\left(\frac{\Delta C_2}{C_2}+\frac{\Delta R_3}{R_3}\right)<M_{н\,\text{доп}}; \tag{2}$$

$$\frac{(\omega_{в}C_0R_{\text{экв}})^2}{(\omega_{в}C_0R_{\text{экв}})^2+1}\left[\frac{R_{\text{экв}}}{R_3}\frac{\Delta R_3}{R_3}+\frac{R_{\text{экв}}}{R_2}\frac{\Delta R_2}{R_2}\right]<M_{в\,\text{доп}}. \tag{3}$$

The condition of minimum cost has the following appearance:

$$\min C=\min\{C_{R_1}(\Delta R_1)+C_{R_2}(\Delta R_2)+C_{R_3}(\Delta R_3)+$$
$$+C_{C_1}(\Delta C_1)+C_{C_2}(\Delta C_2)\}. \tag{4}$$

Depending on the kind of functions $C_{R_1}(\Delta R_1),\ldots,C_{C_2}(\Delta C_2)$ the formulated problem can be solved by various methods.

If all functions are linear, then the problem is resolved by linear programming.

If there is a discrete set of components (in practice each component can be executed with three classes of accuracy, 5, 10 and 20%) and the costs of components are given by a table, then it is a problem of integral nonlinear programming.

Many methods can be used to solve it. First of all, assuming that the number of possible variants is comparatively small, $3^5 = 243$, and calculation of each extremely simple, we can use the method of complete sorting of possible variants.

Setting the variant by accuracy of the components, we test fulfillment of conditions (1)-(3) and calculate C according to formula (4). Calculation stops with nonfulfillment of the first condition. Solution of this problem presents no diffi-

culties, even without the use of computers.

In work [14], a modified method of dynamic programming, more economical than complete sorting, is suggested for solving this problem.

<u>Selection of the optimal diameter of a pipe</u>. Let it be required to determine the optimal diameter of a pipe intended for pumping petroleum using a pump with an electric motor. Without dwelling on the detailed derivation of mathematical dependences, we note only that as the diameter of the pipe D increases, there is a rise in capital expenditures, connected with installation of the pipe, as well as with its amortization and running repair. At the same time, expenditures for electric energy are reduced (its hydraulic resistance is significantly reduced). In summary, these expenditures are determined by the following formula:

$$M = a^*_1 D + a^*_2 D^{-4} + a^*_3 D^{-4,75} + a^*_4. \tag{5}$$

To determine the optimal value of D, the partial derivative of M according to D must be equalized to zero.

Then, after conversions, we obtain

$$a_1 D^{-5} + a_2 D^{-5,75} = 1. \tag{6}$$

This equation is easily solved by the numerical method. In the case of mass calculations it is not complicated to compile a table with two input sections a_1 and a_2. Formulae for calculating these coefficients, obtained on the basis of [45], are given below:

$$a_1 = \frac{0,0103 \rho T v^3 C_{эл}}{\gamma \delta L \eta (1+f)(\alpha + E_н) C_{тр}}; \tag{7}$$

$$a_2 = \frac{0,00065 (\rho\mu)^{0,25} T v^{2,75} C_{эл}}{\gamma \delta \eta (1+f)(\alpha + E_н) C_{тр}}, \tag{8}$$

where μ and ρ — viscosity, $kg{\cdot}sec/m^2$, and specific weight of pumped liquid, kg/m^3; T — fund of work time, *hours*; v — consumption of liquid, m^3/sec; γ — specific weight of metal of the pipe, t/m^3; δ — thickness of the wall of the pipe, *mm*; L — length of the pipeline, *m*; η — efficiency of electric motor; f — relative expenditures for assembly; α — standard of amortization deductions; $E_н$ — standard coefficient of the effectiveness of investments; $C_{эл}$ and $C_{тр}$ — cost of electric energy, rubles/

/kwt-hr, and of pipes, rubles/*kg*.

We note that by analyzing formulae (6)-(8) we can reach qualitative conclusions about the effect of various factors on the optimal diameter of the pipe. It increases with increase of viscosity, specific weight and consumption of liquid, fund of work time and cost of electric energy, and decreases with rise in specific weight, thickness, length and cost of pipes, efficiency of the electric motor, as well as expenditures for assembly of the pipes, stnadards of amortization deductions and standard efficiency coefficient of investments.

<u>Selection of the optimal combination of material, production accuracy and operational loads</u>. Let us be required to select the optimal ultimate strength of material and its spread, optimal margin tolerance Δ and spread of operational loads for pipe with length L, operating under internal pressure $p_{вн}$, if an increase of its weight causes additional expenditures, but reliability P is given.

Turning to formula (6.4.12), we find that

$$\frac{\eta_0 - 1}{\sqrt{\eta_0^2 v_R^2 + v_Э^2}} = A, \qquad (9)$$

where A is determined from condition

$$F_0(A) = P. \qquad (10)$$

In this case

$$R = \frac{2\sigma\delta}{D}; \qquad (11)$$

$$Э = p_{вн}; \qquad (12)$$

$$\eta_0 = \frac{2\sigma_0\delta_0}{D p_{вн}}; \qquad (13)$$

$$v_R = \sqrt{v_\sigma^2 v_\delta^2 + v_\delta^2 + v_\sigma^2} \approx \sqrt{v_\delta^2 + v_\sigma^2}. \qquad (14)$$

We designate the mathematical expectations of values by the index "0." Let also be given the functional connections between parameters of the structure and the cost per unit of weight of the pipe, the spread of pressure and expenditures for its regulation, as well as the weight of the pipe and additional expenditures:

$$C_{\text{уд}} = C_{\text{уд}}(\sigma; v_\sigma; v_\delta); \tag{15}$$

$$C_p = C_p(v_p); \tag{16}$$

$$C_д = C_д(Q). \tag{17}$$

We write the **formula for calculating weight**

$$Q = \gamma\pi DL\delta_{max} = \gamma\pi DL(\delta_0 + \Delta), \tag{18}$$

where γ — specific weight of the material of the pipe; Δ — tolerance for thickness of the pipe wall.

Assuming in first approximation that dimensions in the tolerance zone are distributed according to the law of even probability, we obtain the following connection between v_δ and Δ:

$$v_\delta = \frac{\Delta}{2\sqrt{3}\delta_0}. \tag{19}$$

We now write an equation for the cost of the pipe, the cost of a pressure regulator and additional expenditures connected with the weight of the pipe,

$$C = QC_{\text{уд}}(\sigma; v_\sigma; \Delta) + C_p(v_p) + C_д(Q). \tag{20}$$

Thus, mathematically, the problem is formulated as follows. Determine $\sigma_0, \Delta, v_\sigma$ and v_p, at which C is minimized, calculated according to formula (20) with observance of connections (9), (13), (14), (18) and (19). The latter can be replaced by one equation

$$\frac{\frac{2\sigma_0\delta_0}{Dp_{\text{вн}}} - 1}{\sqrt{\left(\frac{2\sigma_0\delta_0}{Dp_{\text{вн}}}\right)^2\left(\frac{\Delta^2}{12\delta_0^2} + v_\sigma^2\right) + v_p^2}} = A, \tag{21}$$

from which, after uncomplicated operations, we obtain

$$\delta_0 = \frac{Dp_{\text{вн}}}{2\sigma_0(1 - A^2v_\sigma^2)}\left[1 + A\sqrt{v_\sigma^2 + v_p^2 - A^2v_\sigma^2v_p^2 + \frac{1}{3}\left(\frac{\sigma_0\Delta}{Dp_{\text{вн}}}\right)^2(1 - A^2v_p^2)}\right]. \tag{22}$$

One effective means of solving this problem can be the random search method. In a number of cases, as will be seen from the example, effective methods can be regular search, in particular, coordinate search.

Example 7.1.1. For steel pipe, charged with internal pressure $p_{вн} = 5$ кг/мм², length $L = 1{,}000$ *mm* and diameter $D = 100$ *mm*, we must select optimal material σ_0 and spread of its characteristics v_σ, margin tolerance Δ and spread of operational loads v_p, if its reliability is given ($P = 0.95$), and the cost of the device, including pipe and pressure regulator, is determined by formula

$$C=Q\left[0{,}2+2\times 10^{-6}\frac{\sigma_0^{1{,}5}}{\Delta^{0{,}4}v_\sigma^{0{,}5}}\right]+\frac{0{,}2}{v_p}+Q. \tag{23}$$

where σ_0 is expressed in kg/mm², Δ — mm, Q — kg.

Possible limits of change in parameters are the following:

$$v_\sigma = 0{,}02 - 0{,}10;\quad v_p = 0{,}10 - 0{,}30;\quad \Delta = 0{,}1 - 1{,}3 \text{ мм};$$
$$\sigma_0 = 40 - 130 \text{ кг/мм}^2;\quad \delta_0 \geq 3 \text{ мм}.$$

First of all we determine A from equation (10)

$$A = F_0(P) = F_0(0{,}95) = 1{,}645.$$

Then, fixing three out of the four variables (first assuming they are equal to average values of permissible intervals), we calculate δ_0 according to formula (22), Q according to formula (18), and C according to formula (20), covering the entire range of change in the variable. Having found the extremum of C according to this value, we vary the next, etc.

Results of calculations are given in Table 7.1.1.

TABLE 7.1.1

v_σ	v_p	Δ	σ_0	δ_0	Q	C
0,05	0,20	1,3	90	3,9273	12,4807	16,8118
			100	3,5698	11,6271	15,8646
			110	3,2791	10,9331	15,1081
			$\boxed{120}$	3,0392	10,3603	14,4995
			130	2,8373	10,2667	14,5120
0,05	0,20	1,3	$\boxed{120}$	3,0392	10,1215	14,1883
		1,2		3,0082	10,0475	14,1341
		1,1		3,0000	9,7892	13,8425
		1,0		3,0000	9,5504	13,5817
		0,9		3,0000	9,3116	13,3248
		0,8		3,0000	9,0729	13,0778
		0,7		3,0000	8,8341	12,8403
		0,6		3,0000	8,5954	12,6175
		0,5		3,0000	8,3566	12,4134
		0,4		3,0000	8,1178	12,2456
		$\boxed{0,3}$		3,0000	7,8779	12,1409
		0,2		3,0000	7,6403	12,1724
0,5	0,30	$\boxed{0,3}$	120	3,1531	8,2446	12,1676

Continuation of Table 7.1.1.

$v_{\text{ч}}$	$v_{\text{э}}$	Δ	σ_0	b_0	Q	C
	$\boxed{0,25}$			3,000	7,8791	11,9421
	0,20			3,000	7,8791	12,1421
0,05	$\boxed{0,25}$	0,3	120	3,0000	7,8791	12,1426
0,06				3,0007	7,8808	12,0000
0,07				3,0193	7,9252	11,9587
$\boxed{0,08}$				3,0392	7,9727	11,9186
0,09				3,0632	8,0300	11,9208
$\boxed{0,08}$	0,25	0,3	110	3,3141	8,6290	12,4388
			$\boxed{120}$	3,0392	7,9727	11,7186
			130	3,0000	7,8791	11,7614
0,08	0,25	0,3	$\boxed{120}$	3,0392	7,9727	11,8868
		$\boxed{0,2}$		3,0333	7,7198	11,6648
		0,1		3,0000	7,4016	11,8550
0,08	0,30	$\boxed{0,2}$	120	3,1990	8,1155	12,0918
	$\boxed{0,25}$			3,0333	7,7198	11,6648
	0,20			3,0000	7,6403	11,7021
0,08	$\boxed{0,25}$	0,2	120	3,0333	7,7198	11,6648
$\boxed{0,09}$				3,0572	7,7769	11,6542
0,10				3,0836	7,8399	11,6630
$\boxed{0,09}$	0,25	0,3	120	3,0666	8,0381	11,7551
		$\boxed{0,2}$		3,0572	7,7769	11,6542
		0,1		3,0568	7,5372	11,9324
0,09	0,30	$\boxed{0,2}$	120	3,2203	8,1663	12,0677
	0,25			3,0572	7,7769	$\boxed{11,6542}$
	0,20			3,0000	7,6403	11,6636

Thus, the optimal variant is a device with characteristics: $\sigma = 120\ kg/mm^2$; $\Delta = 0,2\ mm$: $v_{\text{ч}} = 0,09$; $v_{\text{э}} = 0,25$.

7.2. Problems of Reserves

Some problems of this class have already been discussed in Chapter 6, for example, the problem of selecting optimal spare parts (ZIP).

Also possible are more complex problems, when spare parts are supplied in several steps, but these problems belong to the area of rational methods of using technical devices.

Let us consider a problem of a slightly different level, namely the problem of rational reserves of fuel in mobile vehicles.

Let us be required to determine the optimal volume of gasoline tanks in a truck. We designate mass fuel consumption per kilometer as q, the ratio of the weight of the gasoline tank to the weight of the fuel in it as α. Then the summary weight of the fuel and the gas tank is $nq(1 + \alpha)$, where n is the fuel reserve, in km. We designate the time necessary for refuelling as $t_з$, average speed v_{cp} (taking into account idle time, but not refuelling), duration of the work day t_c. Finally, let the load capacity of the truck (in which we must also include the weight of the gas tanks) be Q and its cost (except gas tanks) be C.

The ratio of usefully employed time to time taking into account refuelling takes the form

$$\frac{\beta \frac{n}{v_{cp}}}{\beta \frac{n}{v_{cp}} + t_з} = \frac{1}{1 + \frac{v_{cp} t_з}{\beta n}},$$

where β — coefficient, considering the impossibility of complete fuel consumption. It depends on the network of gasoline pumps.

The number of ton-kilometers per day is

$$t_c v_{cp} [Q - nq(1 + \alpha)] \frac{1}{1 + \frac{v_{cp} t_з}{\beta n}},$$

and expenditures for them are

$$C_э t_з + \frac{t_c v_{cp}}{1 + \frac{v_{cp} t_з}{\beta n}} \left[C_l + \frac{C + C_б (1 + \alpha) nq}{L} \right],$$

where $C_э$ — expenditures connected with the operating time of the truck (the driver's pay, maintenance of the garage, etc.); C_l — expenditures connected

with kilometer run (fuel, oil, repair, etc.); $C_{\text{б}}$ — cost of a kilogram of gas tank and reinforcement for its installation; L — lifetime of truck, *km*.

Then the cost of a ton-kilometer is

$$C_{\text{т}}=\frac{\frac{C_0}{v_{\text{ср}}}\left(1+\frac{v_{\text{ср}}t_0}{\beta n}\right)+\left[C_l+\frac{C+C_{\text{б}}(1+\alpha)nq}{L}\right]}{[Q-nq(1+\alpha)]}. \qquad (1)$$

This expression can be represented as follows:

$$C_{\text{т}}=\frac{a+bn+\frac{d}{n}}{1-en}, \qquad (2)$$

where

$$a=\frac{1}{Q}\left(C_l+\frac{C}{L}+\frac{C_0}{v_{\text{ср}}}\right); \qquad (3)$$

$$b=\frac{C_{\text{б}}(1+\alpha)}{LQ}q; \qquad (4)$$

$$d=\frac{C_0 t_0}{\beta Q}; \qquad (5)$$

$$e=\frac{q(1+\alpha)}{Q}. \qquad (6)$$

We must find n, minimizing $C_{\text{т}}$, for which we must take a partial derivative according to n and equalize it to zero.

After uncomplicated operations, assuming that $n > 0$, we obtain

$$n=\frac{de}{b+ae}\left[\sqrt{1+\frac{b+ae}{de^2}}-1\right]. \qquad (7)$$

As $\frac{b+ae}{de^2}\gg 1$, then we can write

$$n\approx\sqrt{\frac{d}{b+ae}}. \qquad (8)$$

Example 7.2.1. Let q = 0.20 *kg/km*; α = 2; β = 0.5; t_0 = 0.5 hrs; $v_{\text{ср}}$ = 20 *km/hr*; Q = 2000 *kg*; C = 5000 rubles; C_0 = 1 rub/*hr*; $C_{\text{б}}$ = 1 rub/*kg*, C_l = 0.025 rub/*km*; L = 200,000 *km*.

Determine the optimal fuel distance n. We calculate:

$$a=\frac{1}{2000}\left(0{,}025+\frac{5000}{200\,000}+\frac{1}{20}\right)=0{,}05\times10^{-3};$$

$$b=\frac{1\,(1+2)\times0{,}20}{200\,000\times2\,000}=0{,}15\times10^{-8};$$

$$d=\frac{1\times0{,}5}{0{,}5\times2000}=0{,}5\times10^{-3};$$

$$e=\frac{0{,}20\,(1+2)}{2000}=0{,}3\times10^{-3};$$

$$n=\sqrt{\frac{0{,}5\times10^{-3}}{0{,}15\times10^{-8}+0{,}05\times10^{-3}\times0{,}3\times10^{-3}}}=170\ \text{км}.$$

In practice, considering the desirability of certain reserves, fuel distance can be assumed to be slightly higher.

7.3. Problems Connected with the Replacement of Equipment

A generalized problem of the replacement of equipment was considered in Chapter 4; however, in practice we meet a very wide range of varied problems concerning the replacement of equipment. One of these problems was duscussed in Chapter 6.

As one more example of this class of problems, let us consider determination of the economic lifetime of a technical device, using materials from [47, 27].

In designing a new technical device, it is necessary to determine its efficient service life so that its basic units and assemblies are designed for this particular length of time.

First of all, we must distinguish physical service time $T_{ф}$, at the end of which the technical device is unfit for operation and not repairable; economic service life $T_{эк}$, guaranteeing minimal expenditures for operation, including capital expenditures; and efficient service life $T_{рац}$, considering besides economy, the actual possibilities of the state for renovating a fleet of technical devices (industrial capabilities, materials, etc.).

Obviously, efficient service time occupies an intermediate position between physical and economic times. In designing, there should be an attempt made to coincide physical and efficient times, as increase of physical operating time is

connected with a rise in the cost of the technical device.

Let us consider now the cost of operating a technical device for a year, including capital expenditures (cost of buying the technical device $C_{\text{н}}$),

$$C = A + C_{\text{р}} + \frac{C_{\text{н}} - C_{\text{ир}}}{T}, \tag{1}$$

where A — fixed annual expenditures, not depending on times of operation (for a truck these are expenditures for the driver's pay, fuel, oil and tires); $C_{\text{р}}$ — expenses for technical service and repair per year (for a truck, this is the cost of technical service, medium and major repairs); $C_{\text{ир}}$ — value returned when the technical device is given up (for example, trucks into scrap metal); T — lifetime.

Analysis of statistical data for a great variety of technical devices shows that $C_{\text{р}}$ is a complex function of time, which in first approximation can be approximated by a linear function

$$C_{\text{р}} = a + bt, \tag{2}$$

where coefficients a and b depend on the type of technical device and can be determined by analogy with existing ones. This function by analogy with the function of intensity of breakdown resembles the one shown in Fig. 7.3.1, but for practical purposes, we can also use linear dependence, as actually economic service time is selected in a linear segment.

We determine economic service time, based on the minimum of average annual expenditures during the entire time of operation. Evidently, taking into account (1) and (2)

$$C_{\text{ср}} = A + a + \frac{b}{2}T + \frac{C_{\text{н}} - C_{\text{ир}}}{T}. \tag{3}$$

Taking the derivative of $C_{\text{ср}}$ according to T and equalizing it to zero, after simple conversions, we obtain

$$T_{\text{эк}} = \sqrt{2\frac{C_{\text{н}} - C_{\text{ир}}}{b}}. \tag{4}$$

However, we must consider that technical devices are subject not only to physical aging, but also to obsolescence.

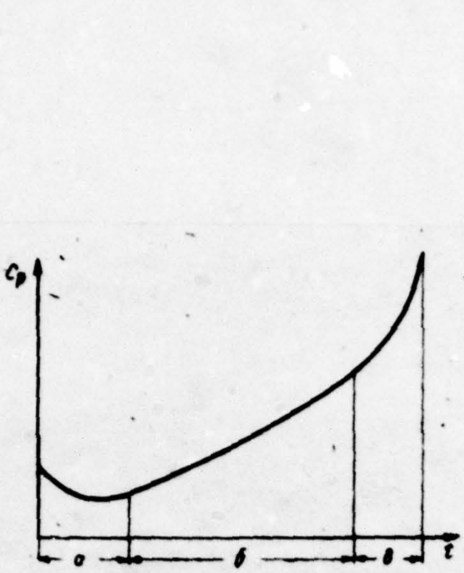

Figure 7.3.1. Change in expenses for operation in proportion to the service of the technical device:

a — lapped segment; b — section of normal operation; c — section of heavy wear

K. Marx writes: "But besides material wear, a machine is also subject to obsolescence, so to speak. It loses its exchange value as machines of the same design begin to be produced more cheaply or better machines come into competition with it."[1]

Thus, a reduction in the cost of the same type of technical device or the appearance of new, improved technical devices can reduce economic service time. We give an approximate evaluation of this situation.

Let with the course of time production of a technical device become cheaper, for example, according to law

$$C_{п} = C_{п_0} - kt^2. \qquad (5)$$

This causes an additional loss of value of the current technical device, which in time T is kT^2. Taking this into account, we write equation (3) as follows:

$$C_{ср} = A + a + \frac{b}{2}T + \frac{C_{п} - C_{лв}}{T} + kT. \qquad (6)$$

[1]K. Marx, F. Engels. Works. 2nd ed., Vol. 23, p. 415.

From which

$$T_{эк}=\sqrt{2\frac{C_{в}-C_{вр}}{b+2k}}. \tag{7}$$

i.e. less than in the preceding case.

Having determined economic service time, on the basis of the demand for this kind of device, we must calculate the production capacity and consumption of materials. Here we must keep in mind the following dependences. If the demand for annual production of devices for development of the economy is $\varphi(t)$, then the necessary number of technical devices to be produced, taking into account the necessity of replacing those which have worn out, is

$$F(t)=\begin{cases} \varphi(t) & \text{при } 0<t<T_{рац}; \\ \varphi(t)+\varphi(t-T_{рац}) & \text{при } T_{рац}<t<2T_{рац}; \\ \varphi(t)+\varphi(t-T_{рац})+\varphi(t-2T_{рац}) & \\ & \text{при } 2T_{рац}<t<3T_{рац}. \end{cases} \tag{8}$$

[при = with]

Example 7.3.1. Determine the economic service time of a vehicle if it is known that its purchase price, taking into account return value $(C_{в}-C_{вр})$, is 1000 units, and the increase of operational expenses b is 20 units per year.

According to (4)

$$T_{эк}=\sqrt{\frac{2\times 1000}{20}}=10 \text{ years.}$$

Example 7.3.2. Solve the same problem with the condition that production of vehicles is 50% cheaper in 10 years. Then according to (5) and (7)

$$k=\frac{1000\times 0,5}{10^{2}}=5;$$

$$T_{эк}=\sqrt{\frac{2\times 1000}{20+5\times 2}}=8,2 \text{ years.}$$

i.e. less than in the preceding case.

Example 7.3.3. Let the demand for vehicles of the discussed class increase with time according to linear law $N = 100\ t$, i.e. the annual production of vehicles, without taking into account replacement of those which wear out, is 100.

We must calculate the necessary volume of production at various moments of time with T_{se} = 5, 10 and 15 years. We make calculations according to formula (8); results are given in Table 7.3.1.

TABLE 7.3.1.

T_{se}, yr	t, year					
	5	10	15	20	25	30
5	100	200	300	400	500	600
10	100	100	200	200	300	300
15	100	100	100	200	200	200

As can be seen from Table 7.3.1, the increase of production connected with replacing broken down vehicles is very significant and approximately proportional to operational times. We cannotignore this in the final selection of efficient service times for technical devices.

7.4. Mass Service Problems

Many technical devices can be considered mass service systems, i.e. systems in which claims for service appear at random moments of time and service takes random segments of time. By analyzing such systems using the theory of mass service we can determine the probability that the system will be busy.

In relation to operations research, these problems have much in common with problems of selecting optimal reliability, only instead of the probability of failure, here we figure the probability that the system will be busy (i.e. refusal or delay of service) and instead of optimal reliability we select the optimal throughput of a technical device.

Without dwelling even on a summary account of the principles of mass service theory, we shall present several examples of the solution of technical problems in operations research relating to the theory of mass service, assuming that, if necessary, the reader will acquaint himself with at least one of the books dealing with this theory, for example [36] or [44].

Example 7.4.1. We must determine the optimal number of berths for a port intended to receive perishable cargo (if the berth is occupied when the ship arrives, the cargo will spoil). Such systems of mass servicing, in which the claim cannot wait, are called a system with failures. The flow of ships is very simple and is characterized by density $\lambda = 0.5$ *ships/day*. Unloading time has exponential distribution, averaging $\bar{t}_{обс} = 2$ days. The value of the cargo $C_2 = 100{,}000$ rubles, cost of operation of the berth per day, including capital expenditures, is $C_1 = 10{,}000$ rubles.

For this case, the probability of a failure in servicing is

$$P_n = \frac{\frac{a^n}{n!}}{\sum_{k=0}^{n} \frac{a^k}{k!}}, \qquad (1)$$

where

$$a = \lambda \bar{t}_{обс};$$

n — number of devices (lines, berths). The mathematical expectation of daily expenditures is

$$M = nC_1 + P_nC_2. \qquad (2)$$

Results of calculations are given in Table 7.4.1.

TABLE 7.4.1.

$a = 0{,}5 \times 2 = 1$

Number of berths	P_n	M
1	0,500	60 000
2	0,200	40 000
3	0,062	36 200
4	0,015	41 500
5	0,003	50 300

From the data in the Table it can be seen that the optimal number of berths is 3.

Example 7.4.2. A computer consists of a system for storing incoming data (ЗУ) (volume n messages) and a data processing system (ОИ), which processes messages transmitted from the ЗУ in groups in average time $\bar{t}_{обс}$. After processing all incoming messages, the processing unit selects from the ЗУ all data accumulated in this time. When the ЗУ is filled, incoming data is lost and expenditures for repeat transmission are $C_2 = 0.002$.

The density of data input $\lambda = 1$ *message/min*. We must find optimal n and $\bar{t}_{обс}$ if it is known that the cost of ЗУ and ОИ is

$$C_{ЗУ} = k_{ЗУ} n = 180\, n; \tag{3}$$

$$C_{ОИ} = \frac{k_{ОИ}}{\bar{t}_{обс}} = \frac{750}{\bar{t}_{обс}}. \tag{4}$$

They are calculated on operation for $T = 500{,}000$ *min* $\approx$ 1 year. This system, which is called a system with a bunker, has dependence

$$P_{отк} = \left(\frac{\alpha}{\alpha+1}\right)^n. \tag{5}$$

We can write the expression for the mathematical expectation of expenditures for T years (assuming that operational expenditures do not depend on n or $\bar{t}_{обс}$ and therefore disregarding them):

$$M = P_{отк} C_2 \lambda T + K_{ЗУ} n + \frac{K_{ОИ}}{\bar{t}_{обс}}. \tag{6}$$

Substituting (5) in (6), after conversions, we obtain

$$M = \left(\frac{\alpha}{\alpha+1}\right)^n C_2 \lambda T + k_{ЗУ} n + \frac{K_{ОИ}\lambda}{\alpha}. \tag{7}$$

We must find maximum M according to two variables α and n. Unfortunately, analytical expressions for optimal α and n

cannot be found; therefore, the problem must be solved numerically.

We set n and find minimum M according to α, then we set the next n, etc. Results of calculations of M are given in Table 7.4.2.

TABLE 7.4.2

n	α							
	1	2	3	4	5	6	7	8
1	1 400	1 192	1 150	1 138	1 134	1 132	1 132	1 133
2	1 300	1 119	1 112	1 128				
3	1 325	1 121	1 122					

From the data in Table 7.4.2 it can be seen that the optimal variant is one with $n = 2$ and $\alpha = 3$, i.e. $i_{obc} = 3$.

Example 7.4.3. Let us consider the conditions of Example 7.4.1, but with one difference: from arrival until the start of unloading, the cargo can wait $t = 2$ days, otherwise it loses its quality. This is a system with a limited waiting time. In this case, the probability that time waiting in line τ is greater than t is

$$P(\tau > t) = P_t = \frac{\alpha^n \exp\left[(\alpha - n)\frac{t}{\bar{t}_{obc}}\right]}{(n-1)!\,(n-\alpha)\left(\sum_{k=0}^{n-1} \frac{\lambda}{k!}\right) + \alpha^n}. \tag{8}$$

The equation of the mathematical expectation of daily expenditures by analogy with (2) takes the form

$$M = nC_1 + P_tC_2. \tag{9}$$

Results of calculations with various n, using this equation, are given in Table 7.4.3.

From Table 7.4.3 it can be seen that the optimal number of n in this case is also 3, but expenditures M are markedly reduced.

TABLE 7.4.3

n	P_t	M
2	0,184	38 400
3	0,022	32 200
4	0,002	40 200

Having established the connection between t (i.e. time of storing the cargo) and the cost necessary for this equipment, we can also solve a more complicated problem — not only selecting the optimal number of berths, but also optimal characteristics of equipment for storing the load.

Example 7.4.4. An electronic computer can be arbitrarily be divided into two parts: the input unit and all the rest. Known is the flow of demands, incoming to the input unit ($\lambda = 5$ *demands/min*). If the input unit is occupied, then the demands wait until it is free. After servicing in the first phase, they enter the machine (second phase) where they are serviced in the same order. The functional connections between average times of servicing demands in the phases and their cost (C_1 and C_2) are known. We must determine optimal servicing time in the phases, i.e. those times with which for a fixed cost of the machine C occurs its maximum throughput (i.e. productivity).

This system is a two-phase mass servicing system with unlimited waiting time.

For it there is a dependence which connects the mathematical expectation of the number of demands made on the system with its characteristics:

$$M = \frac{\lambda \bar{t}_{обс_1}}{1 - \lambda \bar{t}_{обс_1}} + \frac{\lambda \bar{t}_{обс_2}}{1 - \lambda \bar{t}_{обс_2}}. \quad (10)$$

Let

$$C_1 = \frac{k_1}{\bar{t}_{обс_1}}; \quad C_2 = \frac{k_2}{\bar{t}_{обс_2}} \quad (11)$$

and, in addition $C_1 + C_2 = C$. (12)

Thus, the problem is formulated mathematically as follows. Minimize M, determined by formula (10), with observance of condition (12) and connection (11).

After corresponding substitutions and conversions we obtain

$$M = \frac{\lambda \bar{t}_{обс_1}}{1 - \lambda \bar{t}_{обс_1}} + \frac{\dfrac{\lambda k_2}{C - \dfrac{k_1}{\bar{t}_{обс_1}}}}{1 - \dfrac{\lambda k_2}{C - \dfrac{k_1}{\bar{t}_{обс_1}}}} \tag{13}$$

or

$$\frac{M}{\lambda} = \frac{\bar{t}^2_{обс_1}(C - 2\lambda k_2) - \bar{t}_{обс_1}(k_1 - k_2)}{(1 - \lambda \bar{t}_{обс_1})\,[(C - \lambda k_2)\,\bar{t}_{обс_1} - k_1]}. \tag{14}$$

Having taken the partial derivative of M according to $\bar{t}_{обс_1}$, and having equalized it to zero, after rather cumbersome operations we obtain an expression for the optimal value of $\bar{t}_{обс_1}$:

$$\bar{t}_{обс_1} = \frac{\sqrt{k_1 k_2} + k_1}{C - \lambda k_2 + \lambda\sqrt{k_1 k_2}}. \tag{15}$$

Let C = 20,000 rubles; k_1 = 2000 rub/min; k_2 = 1000 rub/min. Then

$$\bar{t}_{обс_1} = \frac{\sqrt{2000 \times 1000} + 2000}{20000 - 5 \times 1000 + 5\sqrt{2000 \times 1000}} = 0{,}154 \text{ min.}$$

$$\bar{t}_{обс_2} = \frac{k_2 \bar{t}_{обс_1}}{C\bar{t}_{обс_1} - k_1} = \frac{1000 \times 0{,}154}{20000 \times 0{,}154 - 2000} = 0{,}143 \text{ min.}$$

i.e. servicing times in both phases must be relatively close.

7.5. Queuing Problems

The problems of selecting the optimal (in some sense or other) sequence of performing operations are usually called queuing problems.

The classic queuing problem is formulated as follows. There are several parts, each of which must be machined in several devices. Given are times of machining

and the sequence of machining each part in each device. We must select the order of machining parts so that the total time of machining the parts will be minimal. It is important to point out that here we could be speaking not only of machining parts, but also of processing information and, in general, any operations.

We can encounter problems of this class in designing technical devices connected with processing dissimilar parts, information, etc.

No general algorithm has been created for solving these problems. There are algorithms for solving a problem with no more than three devices [18]. These problems sometimes are called scheduling problems.

The second, currently most developed variety of queuing problems — network planning and control [30] — is finding greatest application as tactical problems. The essence of problems of this group is the following. It is necessary to perform a certain number of elementary operations, part of which can be performed only sequentially, and part — in parallel. Known are the times for performing each operation (usually these are random values). This sequence is usually in the form of a network (graph), and as a result, these methods are also called network planning methods. In the simplest case the problem consists of finding the time to perform all operations, for which first is determined the "critical path," i.e. the sequence of operations which limits the total time of performing operations, and on this path is computed time.

In the more common case the problem of analyzing a network graph amounts to finding the duration of operations, optimal in some sense (for example, minimizing expenditures for performance of all operations).

In an even more general case, the structure of the process itself (i.e. the network) can also be corrected.

Network planning methods are used not only in organizing the development of complex designs and their controls, but also in the very process of designing certain kinds of technical devices, primarily any kind of automated equipment (for example, automatically controlled machines).

Actually, in designing automated machines the question arises of creating a flow chart guaranteeing minimum cost of manufactured parts.

The simplest chart of an automatically controlled machine specifies sequential performance of operations in processing a part in both working and idling movements of the machine, including the loading of materials, inspection of articles, cleaning of the machine (losses of the first type); changing, mounting and regulating instruments, delivery of instruments, waiting for the adjustor, correcting the instruments (losses of the second type); regulation and repair of mechanisms of the machine (losses of the third type); nonproductive expenditures of time (losses of the fourth type); losses connected with flaws (losses of the fifth type); readjustment for production of other parts (losses of the sixth type).

In analyzing such a simple chart, many problems arise concerning finding optimal solutions (selection of optimal instruments, optimal cutting regime, optimal control regime, servicing of the machine, etc.); however, these problems are not queuing problems.

Productivity of a machine can be increased most by conducting part of the active and [illegible] operations in parallel. Then the work cycle of the automated machine [illegible] form of a network graph, analysis of which in the case of [illegible] presents certain difficulties, especially if we take into [illegible] of performing individual operations.

Under given conditions such analysis consists of determining the critical path and the corresponding time of the cycle and finding ways of shortening this time, taking into account the economic effectiveness of the considered paths.

Determining for each variant the cost of the machine, of instruments, service and time of processing one part, we can select the optimal variant, which guarantees a minimum of total expenditures for the production of one part.

All this also relates to designing automatic assembly lines, as well as other devices, in which are carried out a rather large number of operations, connected with a certain sequence (or variants of sequences).

7.6. Search Problems

Problems of finding optimal search procedures have arisen in various areas, and no theory has yet been created connecting all groups of these problems. The

first group of search problems [54] is connected with evaluating current production; these are problems of finding optimal statistical methods of controlling production quality (search for defective articles). The second group of problems are problems arising in searching for a required object in space with minimal expenditures of resources [23]. The third group of problems includes problems arising in searching for trouble in complex technical devices, again with a minimal expenditure of resources.

It is interesting to note a certain similarity of these problems with problems of searching for extremums of functions (functionals).

In the first two groups we consider primarily problems of optimal methods of using technical devices, in the third group — basically technical problems [38].

Let there be a complex technical device which consists of a number of units, and each unit — of components. We must find the optimal procedure of finding trouble, assuming there is no information on "symptoms of failure" (i.e. on the possibility, judging by the nature of the failure, of making an assumption about the nature of the failure). The procedure of searching for a failure in this case should be divided into two steps: search for the defective unit and then search for the defective component in the unit.

Let unit r be tested first, and then if no defect is noted, unit s. As a criterion we use the time of finding the defect. Evidently, the optimal search strategy must meet condition

$$T(r,s)-T(s,r)<0, \tag{1}$$

i.e. the time spent in searching for the failure with optimal search strategy should be less at each step than with nonoptimal strategy.

Analysis of this dependence, conducted in [38], shows that it can be replaced by equivalent dependence

$$A_r<A_s, \tag{2}$$

where

$$A_r=\frac{t_r+(1-\delta_r)\left\{T_r+\sum_{i=1}^{n(r)}[t_{r_i}+(1-\delta_{r_i})R_{r_i}]+t_r\right\}}{P_r}; \tag{3}$$

$$A_s=\frac{t_s+(1-\delta_s)\left\{T_s+\sum_{i=1}^{n(s)}[t_{s_i}+(1-\delta_{s_i})R_{s_i}]+t_s\right\}}{P_s}; \tag{4}$$

t_r — time necessary for testing rth (sth) unit; δ — probability that in testing rth (sth) unit will not be found defective, if it actually is not defective; T_r — time of preparing the unit for testing; t_{r_i} — time necessary for testing r_i-component; δ_{r_i} — probability that r_i-th component will not be found defective under the condition that it is, in fact, not defective; R_{r_i} — time necessary to repair or replace r_i- component and repeat testing; P_r — probability that the defect is contained in rth unit.

From that, in the first step, optimal strategy consists of first testing the unit for which the value A_r is minimal. An approximate optimal strategy of search consists of arranging A in order of increase and testing units in this sequence. Strictly speaking, according to the results of each test, we must calculate a new value of P_r and new values of A; however, in practice, we can limit ourselves to calculation of P_r before tests:

$$P_r = \frac{\sum_{i=1}^{n(r)} \lambda_{r_i}}{\sum_{r=1}^{N} \sum_{i=1}^{n(r)} \lambda_{r_i}}, \tag{5}$$

where λ_{r_i} is the intensity of breakdowns of r_i- component.

Optimal search stragety in the second step, i.e. search for the defective component within the unit, consists of initial testing of the component in which the value of

$$a_{r_i} = \frac{t_{r_i} + (1 - \delta_{r_i}) R'_{r_i}}{P_{r_i}} \tag{6}$$

is minimal.

Here

$$P_{r_i} \approx \frac{\lambda_{r_i}}{\sum_{i=1}^{n(r)} \lambda_{r_i}}, \tag{7}$$

R'_{r_i} — time used to repair or replace r_i- component and to repeat test the unit.

There are also other approaches to finding the optimal procedure of searching for a defect, based on use of information theory and considering the cost of steps of testing, as well as information on "symptoms of failure."

7.7. Problems of Selecting the Optimal Path

Problems of this class are distinguished by great variety. They can be arbitrarily divided into several groups. The first two groups of this class are basically tactical problems. This is the problem of the traveling salesman and problems of selecting the optimal route in a specific locality, in a specific sea or air region.

The first problem [6] consists of finding the order of visiting a given set of points, minimizing total expenditures. Expenditures for moving between any two points are known. There are no effective algorithms for strict solution of this problem.

Problems of selecting the optimal route primarily bear a stochastic character and are solved by comparing routes selected according to heuristic rules in statistical models. Problems of these groups presuppose the existence of fixed technical devices.

The third group of problems in this class — problems of selecting optimal trajectories of aircraft movement [25, 39] during the design period — is primarily technical.

These problems are connected with the development of rocket technology and they are dealt with in an extensive literature. They can arbitrarily be divided into the following subgroups:

— problems of selecting optimal vertical lift regime of a rocket (for example, to a given altitude with minimum fuel consumption);

— problems of selecting optimal trajectories of ballistic rockets (for example, maximum range with fixed fuel reserves and total weight);

— problems of the optimal movement of satellites from orbit to orbit around a planet or from orbit around one planet to orbit around another.

Without dwelling even on a brief description of methods of solving problems of the third group (they are, as a rule, very complicated), let us note that the statements of many of these problems lack an operational approach (minimum fuel consumption or minimum flight time or minimum weight is sought without taking into account the mutual connection and mutual effect of these factors on the sum of the entire operation), i.e. optimization is conducted according to partial criteria without consider-

ation of complete cost and effectiveness. Below we give a deliberately simplified example of a problem of this class in order to show the essence of this operational approach.

Let it be necessary during design to select the optimal cruising speed of an airplane $V_в$ intended to fly to distance x, at altitude h, with load capacity $Q_п$, if operational expenditures connected with the duration of flight, the cost of structural elements and basic design characteristics of the airplane are given.

We shall consider the thrust of the engine P to be constant (disregarding sections of take off and landing and change in airplane weight). Then

$$\frac{\dot{G}V_e}{g}=P, \tag{1}$$

where $\dot{G}$ — per-second consumption of fuel; V_e — effective exhaust velocity.

The consumption of fuel per flight is

$$\omega=\dot{G}\frac{x}{v}=\frac{gPx}{V_e v}. \tag{2}$$

The weight equation of the airplane is

$$Q=k_{пл}(k_д P+k_б\omega+\omega+k_к Q_п),$$

where $k_д$ — the ratio of the weight of the engine to thrust; $k_б$ — the ratio of the weight of tanks to the weight of the fuel; $k_к$ — the ratio of the weight of the cabin to the weight of the payload; $k_{пл}$ — the ratio of the weight of the airplane to the weight of the cabin with cargo, fuel tanks and engines.

From which

$$Q_п=\frac{1}{k_к}\left[\frac{Q}{k_{пл}}-k_д P-\omega(k_б+1)\right]=$$
$$=\frac{1}{k_к}\left\{\frac{Q}{k_{пл}}-P\left[k_д+\frac{gx}{vV_e}(k_б+1)\right]\right\}. \tag{3}$$

The cost of a flight is

$$C=C_т\omega+\frac{C_д k_д}{T}P+\frac{C_б k_б}{T}\omega+\frac{C_к}{T}[Q-Q_п-$$
$$-k_д P-(1+k_б)\omega]+C_ч\frac{x}{v},$$

where $C_т, C_д, C_б, C_к, C_ч$ — costs of a kilogram of fuel, engine, tank, design of

the airplane and an hour of flight; T — lifetime of the airplane.

Using formulae (2) and (3), after conversions, we obtain

$$C=P\left\{C_{з}\frac{gx}{V_{0}v}+C_{л}\frac{k_{л}}{T}+C_{э}\frac{k_{э}gx}{TV_{0}v}-C_{п}\left(1-\right.\right.$$
$$\left.\left.-\frac{1}{k_{п}}\right)\left[\frac{k_{л}}{T}+\frac{gx(1+k_{э})}{V_{0}vT}\right]\right\}+$$
$$+Q\frac{C_{п}}{T}\left(1-\frac{1}{k_{п}k_{пн}}\right)+C_{т}\frac{x}{v}. \tag{4}$$

The cost of transporting a kilogram of cargo is

$$C_{1}=\frac{C}{Q_{п}}=$$
$$=\frac{P\left\{C_{з}\frac{gxk_{п}}{V_{0}v}+C_{л}\frac{k_{л}k_{п}}{T}+C_{э}\frac{k_{э}k_{п}gx}{TV_{0}v}-C_{п}(k_{п}-1)\times\right.}{\frac{Q}{k_{пн}}-P\left[k_{л}+\right.}\rightarrow$$
$$\rightarrow\frac{\left.\times\left[\frac{k_{л}}{T}+\frac{gx(1+k_{э})}{V_{0}vT}\right]+Q\frac{C_{п}}{T}\left(k_{п}-\frac{1}{k_{пн}}\right)+C_{т}\frac{x}{v}\right\}}{\left.+\frac{gx}{vV_{0}}(k_{э}+1)\right]}. \tag{5}$$

We now express P in terms of Q and V. To preserve flight speed

$$P=\frac{\rho v^{2}}{2}S_{м}C_{x}, \tag{6}$$

where ρ — density of the air; $S_{м}$ — cross section of the middle of the plane; C_{x} — head drag coefficient.

For subsonic speeds

$$C_{x}=1{,}33\,\mathrm{Re}^{-0,5}\cdot 2k_{1}\frac{S_{кр}}{S_{м}}+1{,}2\cdot 1{,}33\,\mathrm{Re}^{-0,5}k_{2}\frac{S_{бп}}{S_{м}}+$$
$$+\frac{(C_{y}^{кр})^{2}}{\pi\lambda}\frac{S_{кр}}{S_{м}}\alpha^{2}+1{,}2\cdot 2\cdot\alpha^{2}+0{,}1, \tag{7}$$

where Re — Reynolds number; k_1 and k_2 — coefficients calculating the resistance of the shape of the wing and fuselage; $S_{кр}$ and $S_{бп}$ — area of the wing and side surface of the body; α — angle of attack;

$$\lambda=\frac{4l_{кр}^{2}}{S_{кр}}=\frac{4l_{кр}}{b}; \tag{8}$$

b — wing chord; $l_{кр}$ — semispan of wing.

The condition of equality of forces in a vertical plane

$$Q \approx \frac{\rho v^2}{2} S_{\text{м}} \left(C_y^{\alpha} \frac{S_{\text{кр}}}{S_{\text{м}}} + 2 \right) \alpha. \tag{9}$$

Having obtained α from (9), we substitute its value in (7), and then in (6). We calculate that

$$C_y^{\alpha} \frac{S_{\text{кр}}}{S_{\text{м}}} > 2 \text{ and } \frac{(C_y^{\alpha})^2}{\pi\lambda} \frac{S_{\text{кр}}}{S_{\text{м}}} > 2{,}4,$$

we disregard the value of given pressure (0.1) and substitute the value of Re.

After conversions we obtain

$$P = \frac{\rho v^2}{2} S_{\text{м}} \left[2{,}66 \frac{k_1 b^{0.5}}{v^{0.5} \nu^{0.5}} \frac{S_{\text{кр}}}{S_{\text{м}}} + 1{,}596 \frac{k_2 l_{\phi}^{0.5}}{v^{0.5} \nu^{0.5}} \frac{S_{\text{ом}}}{S_{\text{м}}} + \right.$$
$$\left. + \frac{Q^2}{\pi \rho^2 v^4 S_{\text{м}} l_{\text{кр}}} \right]. \tag{10}$$

where ν — coefficient of kinematic viscosity of the air; l_{ϕ} — length of the fuselage.

As the weight of the airplane increases in proportion to $\sqrt[3]{Q}$, its geometric dimensions increase. Having denoted

$$k_b = \frac{b}{\sqrt[3]{Q}}; \; k_{l_{\phi}} = \frac{l_{\phi}}{\sqrt[3]{Q}}; \; k_{l_{\text{кр}}} = \frac{l_{\text{кр}}}{\sqrt[3]{Q}}; \; k_{S_{\text{м}}} = \frac{S_{\text{м}}}{\sqrt[3]{Q^2}}. \tag{11}$$

we obtain the final expression for P:

$$P = \frac{\rho v^{3/2}}{2 \nu^{0.5}} k_{S_{\text{м}}} Q^{5/6} \left[2{,}66 k_1 k_b^{0.5} \frac{S_{\text{кр}}}{S_{\text{м}}} + \right.$$
$$\left. + 1{,}596 k_2 k_{l_{\phi}} \frac{S_{\text{ом}}}{S_{\text{м}}} \right] + \frac{Q^{5/3}}{2 \pi \rho v^2 k_{l_{\phi}}}. \tag{12}$$

To determine optimal speed, we calculate graphs $P = f(v,Q)$, $C_t = f_1(v,Q)$ and $Q_{\text{т}} = f_2(v,Q)$, then determine $C_t = f_3(v)$ with $Q_{\text{т}} = Q_{\text{т max}}$ and find the point of minimum in it.

In order to illustrate the use of variation calculus methods, let us consider a simplified problem of selecting an optimal regime of movement — the problem of selecting the optimal thrust regime of an engine in the case of horizontal flight of an airplane.

In this case we can write the following equation of movement

$$m\frac{dv}{dt}=P-\frac{\rho v^2}{2}S_{\text{м}}C_x, \tag{13}$$

where m — mass of the airplane.

Expression (7) can be written approximately (considering that $\frac{(C_y^{\alpha})^2}{\pi\lambda}\frac{S_{\text{кр}}}{S_{\text{м}}}\gg 2$), as follows:

$$C_x=F_1+\frac{(C_y^{\alpha})^2}{\pi\lambda}\frac{S_{\text{кр}}}{S_{\text{м}}}\alpha^2.$$

Keeping in mind (9) and (8) and assuming that $Q = gm$, we obtain

$$C_x=F_1+\frac{(C_y^{\alpha})^2}{\pi\lambda}\frac{S_{\text{кр}}}{S_{\text{м}}}\frac{Q^2}{\left(\frac{\rho v^2}{2}\right)^2 S_{\text{м}}^2 (C_y^{\alpha})^2\left(\frac{S_{\text{кр}}}{S_{\text{м}}}\right)^2}=$$

$$=F_1+\frac{g^2m^2}{\left(\frac{\rho v^2}{2}\right)^2 S_{\text{м}}\pi 4 l_{\text{кр}}^2}.$$

On the basis of (1) we can write $P=-\frac{dm}{dt}V_e$. Then we write equation (13) as follows:

$$m\frac{dv}{dt}=-\frac{dm}{dt}V_e-\frac{\rho v^2}{2}F_1S_{\text{м}}-\frac{g^2m^2}{\frac{\rho v^2}{2}4\pi l_{\text{кр}}^2} \tag{14}$$

Assuming that path, speed and time are connected by ratio

$$ds=vdt, \tag{15}$$

we obtain after conversions

$$\frac{dv}{ds}=-V_e\frac{1}{m}\frac{dm}{ds}-\frac{1}{2}\rho F_1S_{\text{м}}\frac{v}{m}-\frac{g^2m^2}{2\pi\rho l_{\text{кр}}^2v^3} \tag{16}$$

Introducing dimensionless variables:

$$\bar{m}=\frac{4gm}{V_e^2\rho l_{\text{кр}}\sqrt{\pi F_1S_{\text{м}}}}; \tag{17}$$

$$\bar{x}=\frac{v}{V_e}; \tag{18}$$

$$\bar{s}=s\frac{2g}{l_{\text{кр}}V_e^2}\sqrt{\frac{F_1S_{\text{м}}}{\pi}}; \tag{19}$$

$$\bar{t}=t\frac{2g}{l_{\text{кр}}V_e}\sqrt{\frac{F_1S_{\text{м}}}{\pi}}, \tag{20}$$

we obtain

$$\frac{d\bar{x}}{d\bar{s}}=-\frac{1}{\bar{m}}\frac{d\bar{m}}{d\bar{s}}-\frac{\bar{x}}{\bar{m}}-\frac{\bar{m}}{\bar{x}^3} \tag{21}$$

Based on this equation, we can calculate the path of the airplane as it changes its mass from $\bar{m}_0$ to $\bar{m}_k$:

$$\bar{s}=-\int_{\bar{m}_0}^{\bar{m}_k}\frac{\frac{d\bar{x}}{d\bar{m}}+\frac{1}{\bar{m}}}{\frac{\bar{x}}{\bar{m}}+\frac{\bar{m}}{\bar{x}^3}}\,d\bar{m}=\int_{\bar{m}_0}^{\bar{m}_k}F\,d\bar{m}. \tag{22}$$

We now solve the variation problem of selecting flight conditions at which the extremum of the path is achieved. In this case the subintegral function must be satisfied by an Euler equation (Table 3.3.1), i.e.

$$\frac{\partial F}{\partial \bar{x}}-\frac{d}{d\bar{m}}\left[\frac{\partial F}{\partial\left(\frac{\partial \bar{x}}{\partial \bar{m}}\right)}\right]=0,$$

from which, after uncomplicated conversions, we obtain

$$\bar{m}=\bar{x}^2\sqrt{\frac{\bar{x}+1}{\bar{x}+3}} \tag{23}$$

This ratio should satisfy variable mass and flight speed so that with the given fuel consumption the path achieves maximum. It can be used only when initial speed corresponds to initial mass and other parameters.

With low speeds $\bar{x}\approx 0{,}1$ (i.e. v = 200 *m/sec*) this equation is simplified:

$$\bar{x}=\sqrt{\bar{m}\sqrt{3}}. \tag{24}$$

Putting it into dimensional form, we obtain

$$\frac{v}{V_c}=\sqrt{\frac{4\sqrt{3}\,gm}{V_c^2\rho l_{\text{кр}}\sqrt{\pi F_1 S_{\text{м}}}}}$$

or

$$v=\sqrt{\frac{4\sqrt{3}\,g}{\rho l_{\text{кр}}\sqrt{\pi F_1 S_{\text{м}}}}\,m}. \tag{25}$$

Having substituted (24) in (22), after integration, we obtain

$$\bar{s}=\frac{3\sqrt{3}}{10}\left[\frac{\sqrt{3}}{2}(\bar{m}_0-\bar{m})+2(\sqrt{\bar{m}_0}-\sqrt{\bar{m}})\right]. \tag{26}$$

Example 7.7.1. For an airplane are given: effective exhaust velocity $V_e = 2000$ *m/sec*, semispan of wing $l_{кр} = 6$ м, cross section of midsection $S_м = 1$ м², coefficient $F_1 = 0.3$, initial weight $Q_0 = 5000$ *kg*. Air density at altitude of 5 *km* is $\rho = 0.075$ $kg{\cdot}sec^2/m^4$.

We must find the optimal law of change of speed.

On the basis of (25) we obtain

$$v = \sqrt{\frac{4\sqrt{3}}{0{,}075 \times 6 \times \sqrt{3{,}14 \times 0{,}3 \times 1}}\,Q}.$$

Setting different Q, we calculate v, and then using (26), corresponding $\bar{s}$ and, on the basis of (19), values of s.

Results of corresponding calculations are given in Table 7.7.1.

TABLE 7.7.1

Q, кг	5000	4500	4000	3500	3000
v, м/сек	281,7	267,2	251,9	235,6	218,1
s, км	—	246,9	506,3	779,8	1073,4

[кг = kg; сек = s]

As can be seen from Table 7.7.1, as the airplane moves (reduction of Q), its optimal speed decreases, therefore, we must reduce thrust and per-second consumption.

The above examples in no way exhaust the possibilities of using operations research methods to solve problems arising in the process of designing and analyzing technical devices. They were selected only to show the possibilities of using these methods in the development of technical devices in the most varied areas of technology and at various stages of development.

The creative use of operations research methods will help solve many complex technical problems and ultimately to produce machines, mechanisms and instruments to some degree meeting the criterion of optimalcy (economy, etc.) and thereby achieve economy of forces and means.

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